Chapter 5: Exponential and Logarithmic Functions

5-1 Exponential Functions

Exponential Functions: - a function where the **input (x) is the exponent of a numerical base, a**.

Exponential Function

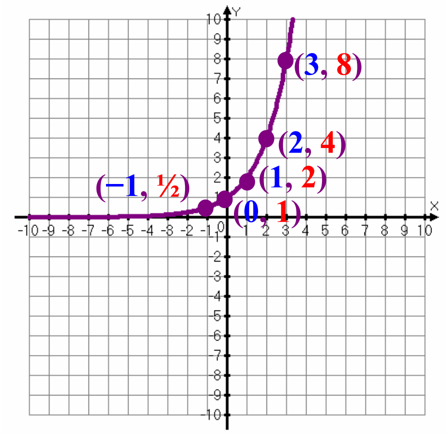
$f(x) = a^x$ **where $a > 0$ and $a \neq 1$**

Note: because $a^0 = 1$, the y-intercept of $f(x) = a^x$ is 1.

Example 1: Graph the following fucntions by creating a small table of values. Generalize yor graph using transformation rules.

a. $f(x) = 2^x$

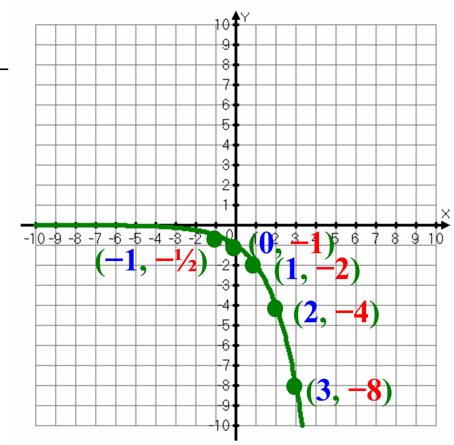
x	$f(x)$
−3	$2^{(-3)} = \frac{1}{8}$
−2	$2^{(-2)} = \frac{1}{4}$
−1	$2^{(-1)} = \frac{1}{2}$
0	$2^{(0)} = 1$
1	$2^{(1)} = 2$
2	$2^{(2)} = 4$
3	$2^{(3)} = 8$



This graph represents the basic exponential function where the **y-intercept is at 1**. The graph **hugs the x-axis on the left side** but **increases drastically as x increases**.

b. $f(x) = -2^x$

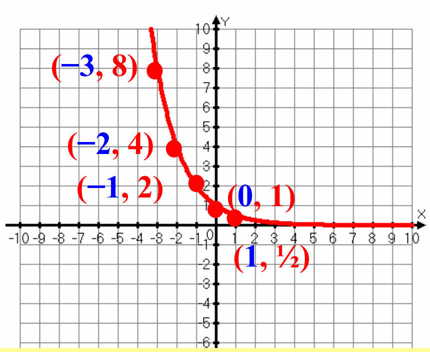
x	$f(x)$
−3	$-2^{(-3)} = -\frac{1}{8}$
−2	$-2^{(-2)} = -\frac{1}{4}$
−1	$-2^{(-1)} = -\frac{1}{2}$
0	$-2^{(0)} = -1$
1	$-2^{(1)} = -2$
2	$-2^{(2)} = -4$
3	$-2^{(3)} = -8$



According to the transformation rules from $y = af(bx + h) + k$, **the graph reflects vertically against the x-axis**. The **y-intercept is reflected to −1**.

c. $f(x) = 2^{-x} = (\frac{1}{2})^x$

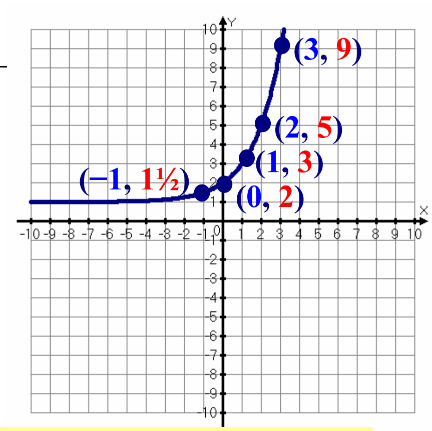
x	$f(x)$
−3	$2^{-(-3)} = 8$
−2	$2^{-(-2)} = 4$
−1	$2^{-(-1)} = 2$
0	$2^{-(0)} = 1$
1	$2^{-(1)} = \frac{1}{2}$
2	$2^{-(2)} = \frac{1}{4}$
3	$2^{-(3)} = \frac{1}{8}$



Using $y = af(bx + h) + k$, **the graph reflects horizontally against the y-axis**. **(Note: a negative exponent can be rewritten where the base, a is between 0 and 1.)** The **y-intercept remains at 1**. The graph **hugs the x-axis on the right side** as it **decreases drastically as x increases**.

d. $f(x) = 2^x + 1$

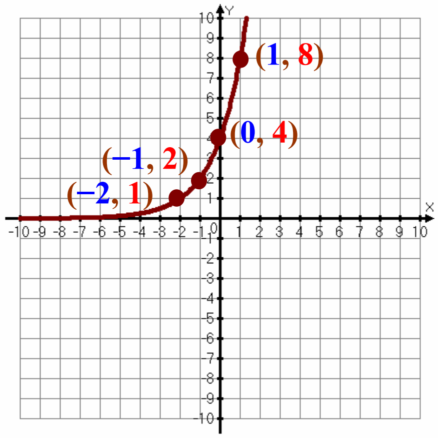
x	$f(x)$
−3	$2^{(-3)} + 1 = 1\frac{1}{8}$
−2	$2^{(-2)} + 1 = 1\frac{1}{4}$
−1	$2^{(-1)} + 1 = 1\frac{1}{2}$
0	$2^{(0)} + 1 = 2$
1	$2^{(1)} + 1 = 3$
2	$2^{(2)} + 1 = 5$
3	$2^{(3)} + 1 = 9$



Using $y = af(bx + h) + k$, **the graph shifted up by 1 unit**. The graph is now **hugging the horizontal line, $y = 1$ on the left hand side**. The **y-intercept is moved to 2**.

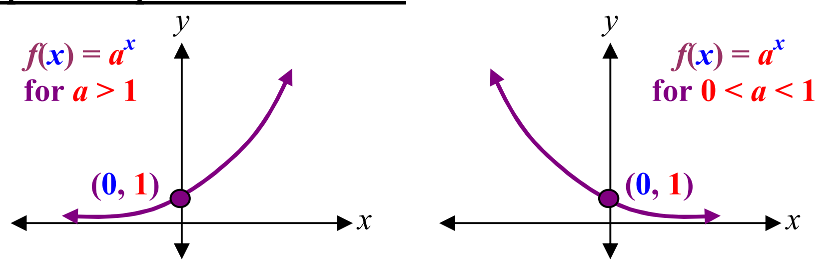
e. $f(x) = 2^{(x+2)}$

x	$f(x)$
−3	$2^{(-3+2)} = ½$
−2	$2^{(-2+2)} = 1$
−1	$2^{(-1+2)} = 2$
0	$2^{(0+2)} = 4$
1	$2^{(1+2)} = 8$
2	$2^{(2+2)} = 16$
3	$2^{(3+2)} = 32$



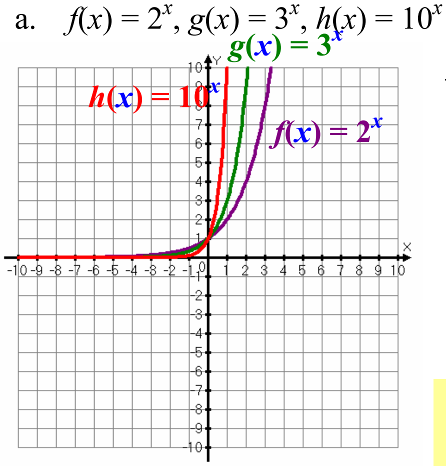
Using $y = af(bx + h) + k$, the graph shifted left by 2 unit. The graph still hugs the x-axis on the left hand side. The y-intercept is now at 4.

Graphs of Exponential Functions



Example 2: Graph the following fucntions by creating a small table of values. Generalize any changes.

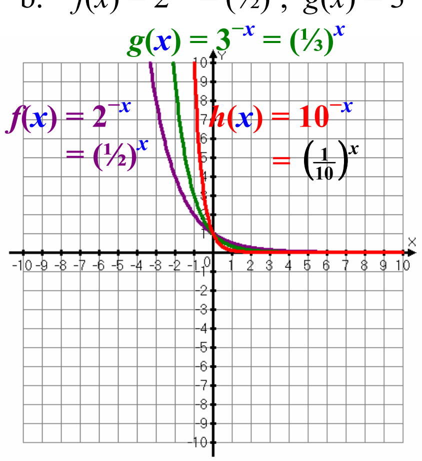
a. $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 10^x$



x	$f(x)$	$g(x)$	$h(x)$
−3	$2^{(-3)} = \frac{1}{8}$	$3^{(-3)} = \frac{1}{27}$	$10^{(-3)} = \frac{1}{1000}$
−2	$2^{(-2)} = \frac{1}{4}$	$3^{(-2)} = \frac{1}{9}$	$10^{(-2)} = \frac{1}{100}$
−1	$2^{(-1)} = \frac{1}{2}$	$3^{(-1)} = \frac{1}{3}$	$10^{(-1)} = \frac{1}{10}$
0	$2^{(0)} = 1$	$3^{(0)} = 1$	$10^{(0)} = 1$
1	$2^{(1)} = 2$	$3^{(1)} = 3$	$10^{(1)} = 10$
2	$2^{(2)} = 4$	$3^{(2)} = 9$	$10^{(2)} = 100$
3	$2^{(3)} = 8$	$3^{(3)} = 27$	$10^{(3)} = 1000$

For bases, $a > 1$, as they increase, the graph increases more sharply as x-increases.

b. $f(x) = 2^{-x} = (½)^x$, $g(x) = 3^{-x} = (⅓)^x$, $h(x) = 10^{-x} = \left(\frac{1}{10}\right)^x$

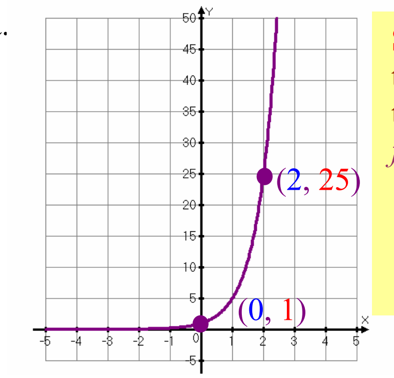


x	$f(x)$	$g(x)$	$h(x)$
−3	$2^{-(-3)} = 8$	$3^{-(-3)} = 27$	$10^{-(-3)} = 1000$
−2	$2^{-(-2)} = 4$	$3^{-(-2)} = 9$	$10^{-(-2)} = 100$
−1	$2^{-(-1)} = 2$	$3^{-(-1)} = 3$	$10^{-(-1)} = 10$
0	$2^{-(0)} = 1$	$3^{-(0)} = 1$	$10^{-(0)} = 1$
1	$2^{-(1)} = \frac{1}{2}$	$3^{-(1)} = \frac{1}{3}$	$10^{-(1)} = \frac{1}{10}$
2	$2^{-(2)} = \frac{1}{4}$	$3^{-(2)} = \frac{1}{9}$	$10^{-(2)} = \frac{1}{100}$
3	$2^{-(3)} = \frac{1}{8}$	$3^{-(3)} = \frac{1}{27}$	$10^{(3)} = \frac{1}{1000}$

For bases, $0 < a < 1$, as they decrease, the graph decreases more sharply as x-increases.

Example 3: Determine the exponential function given the graph below.

a.



Since the y-int = 1, there are no transformations.

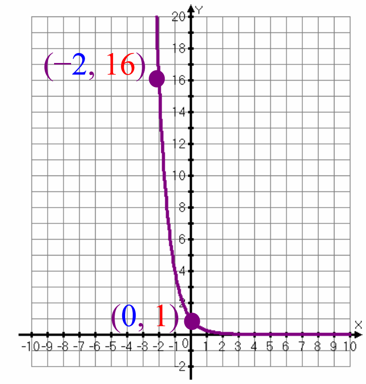
$f(x) = a^x$ $x = 2$, $y = 25$

$25 = a^2$

$a = 5$

$f(x) = 5^x$

b.



Since the y-int = 1, there are no transformations. However, it is decreasing as x increases **($0 < a < 1$)**

$f(x) = a^x$ $x = -2$, $y = 16$

$16 = a^{-2}$

$16 = \frac{1}{a^2}$

$a^2 = \frac{1}{16}$ $a = ¼$

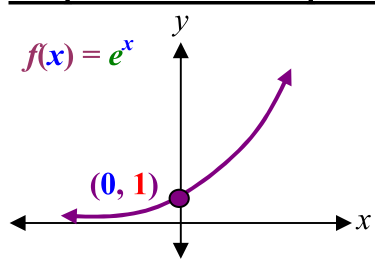
$f(x) = (¼)^x$

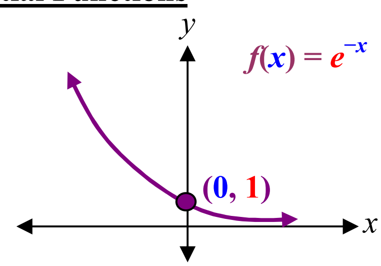
<u>**Natural Exponential Number**</u> **(e)**: - an irrational base number that occurs very frequent in nature (e ≈ **2.71828182845904523…**).

To access the natural number, e, on the graphing calculator, press

2nd **e^x LN** *or* **2nd** **e ÷**

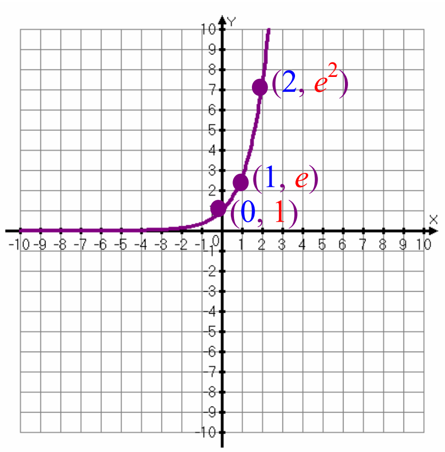
<u>**Graphs of Natural Exponential Functions**</u>





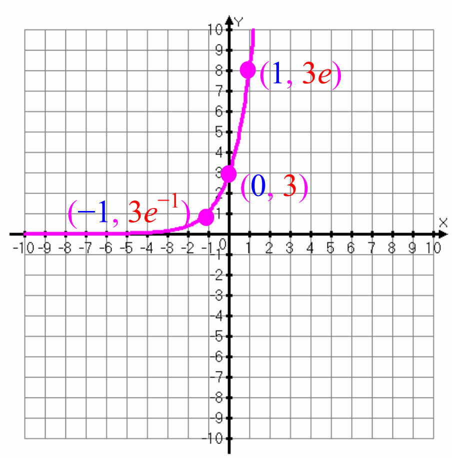
Example 4: Graph the following fucntions. Generalize yor graph using transformation rules.

a. $f(x) = e^x$



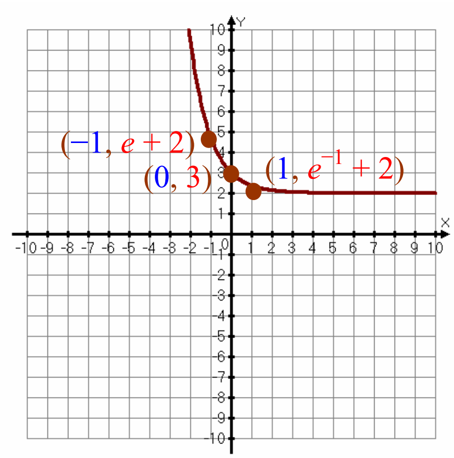
This graph of $y = e^x$ <u>**has the same shape as other exponential graph**</u> where the <u>**y-intercept is at 1**</u>. **The curve exists between the graphs of $y = 2^x$ and $y = 3^x$.**

b. $f(x) = 3e^x$



Using $y = af(bx + h) + k$, <u>**the graph stretched vertically up by a factor of 3**</u>. The graph still <u>**hugs the x-axis on the left hand side**</u>. The <u>**y-interept is stretched up to 3**</u>.

c. $f(x) = e^{-x} + 2$



From $y = af(bx + h) + k$, <u>**the graph reflected horizontally against the y-axis**</u>, and it has <u>**moved up 2 units**</u>. The graph still <u>**hugs the horizontal line of $y = 2$ on the left hand side**</u>. The <u>**y-interept has moved up to 3**</u>.

Compound Interest: - interests earned in every term are not withdrawn, but accumulated.
- the closing balance of each term is the opening balance of the next term.

Term: - the period of time spent before interest is calculated.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = Final Amount after t years **P = Principal**
r = Interest Rate per year **n = Number of Terms per year**

Compound Term per Year (n)	Number of times interest is calculated in a year (nt)	Interest Rate per term ($\frac{r}{n}$) (r = interest rate quoted per annum)
Annually (n = 1)	1	r
Semi-annually (n = 2)	2	$\frac{r}{2}$
Quarterly (n = 4)	4	$\frac{r}{4}$
Monthly (n = 12)	12	$\frac{r}{12}$
Daily (n = 365)	365	$\frac{r}{365}$

Example 5: Mary invested \$2000 compounded semi-annually for 3 years at 4%/a. Using the compound interest formula and the table below, calculate the value of her investment and the total interest earned at the end of the three years.

P = \$2000
n = 2 (semi-annually)
r = 4% / annum (yr) = 0.04
t = 3 years

A = ? **Interest = ?**

$\boldsymbol{A = P\,(1 + \frac{r}{n})^{nt}}$

$A = \$2000(1 + \frac{0.04}{2})^{2(3)}$

$A = \$2000(1.02)^6$

A = \$2252.32

Interest Earned = Final Amount − Principal

Interest = \$2252.32 − \$2000

Interest = \$252.32

Example 6: Using the compound interest formula, calculate the value of her investment and the total interest earned at the end of the three years, if Mary was to invest \$2000 for 3 years at 4%/a

a. compounded quarterly. b. compounded monthly.

P = \$2000
r = 4%/a = 0.04
t = 3 years

A = ?

a. compounded quarterly (n = 4)

$\boldsymbol{A = P\,(1 + \frac{r}{n})^{nt}}$

$A = \$2000(1 + \frac{0.04}{4})^{4(3)}$

$A = \$2000(1.01)^{12}$

A = \$2253.65

b. compounded monthly (n = 12)

$\boldsymbol{A = P\,(1 + \frac{r}{n})^{nt}}$

$A = \$2000(1 + \frac{0.04}{12})^{12(3)}$

$A = \$2000(1 + \frac{0.04}{12})^{36}$

A = \$2254.54

Note: As number of compound period (term) increases, the final amount increases.

5-1 Assignment: pg. 384−388 #15, 17, 19 to 24 (all), 25, 31, 35, 51b, 65, 67, 75
Honours: pg. 384−388 #39 and 69

5-2 Logarithmic Functions

Logarithmic Function: - an **inverse function of an exponential equation**.
- most commonly use to isolate the unknown exponent of an exponential equation.

$$y = a^x \longleftrightarrow x = \log_a y \qquad a > 0 \text{ and } a \neq 1,\quad y > 0$$

Example 1: Convert the following to logarithmic equations.

a. $3^x = 81$

$3^x = 81$

$\log_3 81 = x$

b. $10^{-2} = 0.01$

$10^{-2} = 0.01$

$\log_{10} 0.01 = -2$

c. $x^{-3} = 64$

$x^{-3} = 64$

$\log_x 64 = -3$

d. $\left(\frac{5}{6}\right)^{-3} = \frac{216}{125}$

$\left(\frac{5}{6}\right)^{-3} = \frac{216}{125}$

$\log_{\frac{5}{6}} \frac{216}{125} = -3$

Example 2: Convert the following to exponential equations.

a. $\log_6 36 = x$

$\log_6 36 = x$

$6^x = 36$

b. $-5 = \log_b 32$

$-5 = \log_b 32$

$b^{-5} = 32$

c. $\log_{10} 1000 = 3$

$\log_{10} 1000 = 3$

$10^3 = 1000$

d. $\log_{\sqrt{7}} 49 = y$

$\log_{\sqrt{7}} 49 = y$

$\left(\sqrt{7}\right)^y = 49$

Strategy to Solve Simple Logarithmic Equations

1. If the logarithm is **not in base 10**, **convert it into an exponential form**. **(Note: the log function of all scientific and graphing calculators are in base 10.)**
2. If y is easily recognized as the power of the base, a or some other base, then **write both sides of the exponential equation in the same base**. **Equate the exponents and solve.**

Example 3: Solve the following logarithmic equations.

a. $\log_8 x = 2$

$\log_8 x = 2$

$8^2 = x$

$x = 64$

b. $\log_{\frac{1}{2}} 16 = x$

$\log_{\frac{1}{2}} 16 = x$

$(½)^x = 16$

$(2^{-1})^x = 2^4$

$2^{-x} = 2^4$

$-x = 4$

$x = -4$

c. $\log_x \frac{1}{243} = 5$

$\log_x \frac{1}{243} = 5$

$x^5 = \frac{1}{243}$

$x^5 = \frac{1}{3^5}$

$x^5 = (⅓)^5$

$x = ⅓$

d. $\log_5 5\sqrt{5} = x$

$\log_5 5\sqrt{5} = x$

$5^x = 5\sqrt{5}$

$5^x = (5^1)(5^{½})$

$5^x = 5^{1 + ½}$

$5^x = 5^{3/2}$

$x = \frac{3}{2}$

Simple Properties of Logarithms

$\log_a 1 = 0$	because $a^0 = 1$
$\log_a a = 1$	because $a^1 = a$
$a^{\log_a x} = x$	because exponent and logarithm are inverse of one another
$\log_a a^x = x$	because logarithm and exponent are inverse of one another

Example 4: Evaluate.

a. $\log_3 3$

Let $\log_3 3 = x$
$3^x = 3$
$x = 1$
$\log_3 3 = 1$

b. $\log_2 2^{-6}$

$\underbrace{\log_2 2^{-6}}$
Reverse operation of the same base
$\log_2 2^{-6} = -6$

c. $\log_4 1$

Let $\log_4 1 = x$
$4^x = 1$
$x = 0$
$\log_4 1 = 0$

d. $5^{\log_5 3}$

$\underbrace{5^{\log_5 3}}$
Reverse operation of the same base
$5^{\log_5 3} = 3$

e. $\log_x x^9$

$\underbrace{\log_x x^9}$
Reverse operation of the same base
$\log_x x^9 = 9$

f. $x^{\log_x \frac{2}{3}}$

$\underbrace{x^{\log_x \frac{2}{3}}}$
Reverse operation of the same base
$x^{\log_x \frac{2}{3}} = \frac{2}{3}$

Common Logarithm ($\log_{10} x = \log x$): - a logarithm with a base of 10.

Natural Logarithm ($\log_e x = \ln x$): - a logarithm with a base of the natural number, e.

Common and Natural Logarithm

Common Logarithm:	$\log x = y$	$\longleftrightarrow$	$10^y = x$
Natural Logarithm:	$\ln x = y$	$\longleftrightarrow$	$e^y = x$

Example 5: Evaluate by hand if possible. Otherwise, use a calculator and express the answer to four decimal places.

a. $\log 100$

Let $\log 100 = x$
$10^x = 100$
$x = 2$
$\log 100 = 2$

b. $\log 0.001$

Let $\log 0.001 = x$
$10^x = 0.001$
$10^x = \frac{1}{1000} = \frac{1}{10^3}$
$10^x = 10^{-3}$ $x = -3$
$\log 0.001 = -3$

c. $\ln 1$

Let $\ln 1 = x$
$e^x = 1$
$x = 0$
$\ln 1 = 0$

d. $\ln e$

Let $\ln e = x$ $\rightarrow$ $\log_e e = x$
$e^x = e$
$x = 1$
$\ln e = 1$

e. $\log 250$

We have to use a calculator because 250 is not in base 10.

```
log(250)
      2.397940009
```

2.3979

f. $\ln(4\sqrt{3} + 1)$

We have to use a calculator because $(4\sqrt{3} + 1)$ is not in base 10.

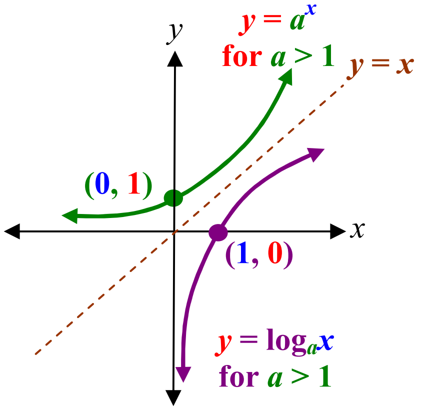
```
ln(4√(3)+1)
      2.070426431
```

2.0704

Graphs of Logarithmic Functions

Since **logarithmic function is an inverse of an exponential function**, we can **reflect** the graph of an exponential function **off the $y = x$ line** to find the graph of a logarithmic function. **The x- and y-values will switch places as well as the domain and range**.

Graphs of Exponential and Logarithmic Functions



Exponential Function
y-int = 1 No x-intercept
Domain $x \in R$; Range $y > 0$

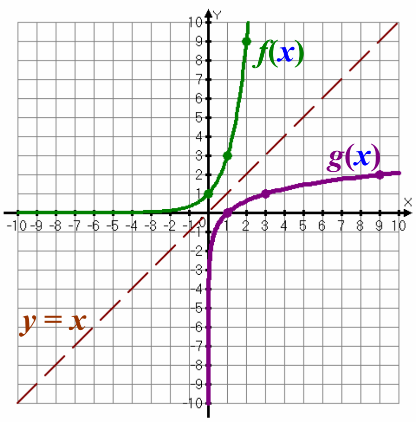
Logarithmic Function
x-int = 1 No y-intercept
Domain $x > 0$; Range $y \in R$

To obtain equation for the inverse of an exponential function, we start with

$y = a^x$
$x = a^y$ (switch x and y for inverse)
$y = \log_a x$ (rearrange to solve for y)

Example 6: Graph the following fucntions using transformation rules. Indicate the domain and range.

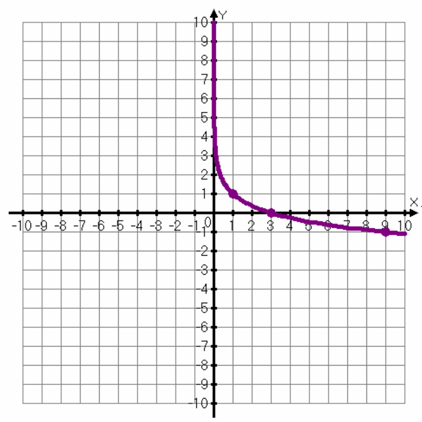
a. $f(x) = 3^x$; $g(x) = \log_3 x$



$g(x) = \log_3 x$ is the inverse of $f(x) = 3^x$

$f(x) = 3^x$	$g(x) = \log_3 x$
(0, 1)	(1, 0)
(1, 3)	(3, 1)
(2, 9)	(9, 2)
Domain	**Domain**
$x \in R$	$x > 0$
Range	**Range**
$y > 0$	$y \in R$

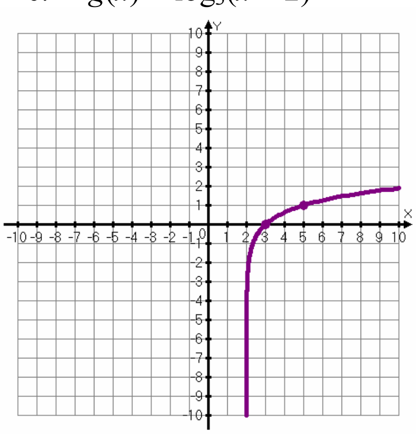
b. $g(x) = -\log_3 x + 1$



Reflect $y = \log_3 x$ on the x-axis and move up 1 unit (neg on y-values then add 1)

$g(x) = -\log_3 x + 1$
$(1, -(0) + 1) = (1, 1)$
$(3, -(1) + 1) = (3, 0)$
$(9, -(2) + 1) = (9, -1)$
Domain: $x > 0$
Range: $y \in R$

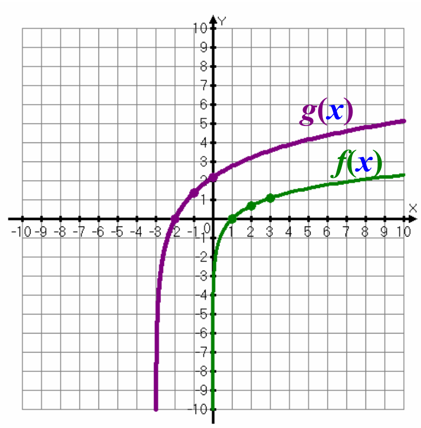
c. $g(x) = \log_3(x - 2)$



Shift $y = \log_3 x$ right by 2 units (take all x-values and add 2)

$g(x) = \log_3(x - 2)$
$((1) + 2, 0) = (3, 0)$
$((3) + 2, 1) = (5, 1)$
$((9) + 2, 2) = (11, 2)$
Domain: $x > 2$
Range: $y \in R$

d. $f(x) = \ln x$; $g(x) = 2\ln(x + 3)$



Shift $y = \ln x$ left by 3 units and vertically stretched by a factor of 2.

$f(x)$	$g(x)$
(1, 0)	(−2, 0)
(2, 0.693)	(−1, 1.386)
(3, 1.099)	(0, 2.197)
Domain	**Domain**
$x > 0$	$x > -3$
Range	**Range**
$y \in R$	$y \in R$

5-2 Assignment: pg. 397−399 #1, 5, 7, 11, 13, 17, 21, 23, 27, 31, 33a, 35a, 39, 41 to 46 (all), 51, 55, 61, 79 Honours: #75

5-3 Laws of Logarithms

Exponential Laws	Logarithmic Laws
$(a^m)(a^n) = a^{m+n}$	$\log_a x + \log_a y = \log_a(xy)$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\log_a x - \log_a y = \log_a\left(\dfrac{x}{y}\right)$
$(a^m)^n = a^{m \times n}$	$\log_a x^y = y \log_a x$
$a^0 = 1$	$\log_a 1 = 0$

Common Logarithm Mistakes

$\log_a(x + y) \neq \log_a x + \log_a y$

Example: $\log(2 + 8) \neq \log 2 + \log 8$
$1 \neq 0.3010 + 0.9031$

$\log_a(x - y) \neq \log_a x - \log_a y$

Example: $\log(120 - 20) \neq \log 120 + \log 20$
$2 \neq 2.0792 + 1.3010$

$\log_a\left(\dfrac{x}{y}\right) \neq \dfrac{\log_a x}{\log_a y}$

Example: $\log\left(\dfrac{1}{10}\right) \neq \dfrac{\log 1}{\log 10}$
$-1 \neq \frac{0}{1}$

$(\log_a x)^y \neq y \log_a x$

Example: $(\log 100)^3 \neq 3 \log 100$
$2^3 \neq 3(2)$

Example 1: Express as a single logarithm. Simplify if possible.

a. $\log_9 3 + \log_9 243$

$\log_9 3 + \log_9 243$
$= \log_9 (3 \times 243)$
$= \log_9 729 \quad \rightarrow \quad 9^x = 729$
$= 3$

b. $\log_2 96 - \log_2 6$

$\log_2 96 - \log_2 6$
$= \log_2\left(\dfrac{96}{6}\right)$
$= \log_2 16 \quad \rightarrow \quad 2^x = 16$
$= 4$

c. $\log 8 + \log 25 - \log 2$

$\log 8 + \log 25 - \log 2$
$= \log\left(\dfrac{8 \times 25}{2}\right)$
$= \log_{10} 100 \quad \rightarrow \quad 10^x = 100$
$= 2$

Example 2: Express as a product. Simplify if possible.

a. $\log_7 49^3$

$\log_7 49^3$
$= 3(\log_7 49) \quad \rightarrow \quad 7^x = 49$
$= 3(2)$
$= 6$

b. $\ln x^z$

$\ln x^z$
$= z(\ln x)$
$= z \ln x$

c. $\log\sqrt{0.0001}$

$\log\sqrt{0.0001}$
$= \log(0.0001)^{½}$
$= ½ \log\left(\dfrac{1}{10000}\right) = ½ \log\left(\dfrac{1}{10^4}\right)$
$= ½ \log 10^{-4}$
$= ½(-4)\log_{10} 10 \quad \rightarrow \quad 10^x = 10$
$= ½(-4)(1)$
$= -2$

Example 3: Express as a single logarithm. Simplify if possible.

a. $\ln x + \ln\sqrt{y} - 3\ln z$

$\ln x + \ln\sqrt{y} - 3\ln z$

$= \ln x + \ln y^{\frac{1}{2}} - \ln z^3$ (change $\sqrt{\ }$ to exponent ½ ; any coefficient goes back as exponent inside the log term.)

$= \mathbf{\ln\left(\frac{xy^{\frac{1}{2}}}{z^3}\right)}$

b. $\log\sqrt[3]{x} - \log y^3 + 2(\log y + \log x^2)$

$\log\sqrt[3]{x} - \log y^3 + 2(\log y + \log x^2)$

$= \log x^{\frac{1}{3}} - \log y^3 + 2\log y + 2\log x^2$

$= \log x^{\frac{1}{3}} - \log y^3 + \log y^2 + \log x^{2\times 2}$

$= \log\left(\frac{x^{\frac{1}{3}}y^2x^4}{y^3}\right) = \mathbf{\log\left(\frac{x^{\frac{13}{3}}}{y}\right)}$

c. $\log_a(x^2 - 5x + 6) - \log_a(x - 3)$

$\log_a(x^2 - 5x + 6) - \log_a(x - 3)$

$= \log_a\left[\frac{(x^2 - 5x + 6)}{(x-3)}\right]$

$= \log_a\left[\frac{(x+2)(x-3)}{(x-3)}\right] = \mathbf{\log_a(x + 2)}$

Note:
$\log_a(x^2 - 5x + 6) \neq \log_a x^2 - \log_a 5x + \log_a 6$

Example 4: Using the Logarithm Laws, expand the expressions below.

a. $\log(y(y - 5))$

$\log(y(y - 5))$

$= \mathbf{\log y + \log(y - 5)}$

b. $\log_4\left(\frac{xy}{z}\right)$

$\log_4\left(\frac{xy}{z}\right) = \mathbf{\log_4 x + \log_4 y - \log_4 z}$

c. $\ln\left(\frac{x^5}{y^7z^2}\right)$

$\ln\left(\frac{x^5}{y^7z^2}\right)$

$= \ln x^5 - \ln y^7 - \ln z^2$

$= \mathbf{5\ln x - 7\ln y - 2\ln z}$

d. $\log_5\sqrt{x^3 + 4}$

$\log_5\sqrt{x^3 + 4}$

$= \log_5(x^3 + 4)^{\frac{1}{2}}$

$= \mathbf{\tfrac{1}{2}\log_5(x^3 + 4)}$

Example 5: Given that $\log 5 \approx 0.699$ and $\log 6 \approx 0.778$, evaluate $\log 150$.

Given: $\log 5 \approx 0.699$ and $\log 6 \approx 0.778$

$\log 150 = \log(5 \times 5 \times 6)$ (Express 150 as factors of 5 × 5 × 6)

$= \log 5 + \log 5 + \log 6$ (Expand using law of logarithm)

$\approx 0.699 + 0.699 + 0.778$

$\mathbf{\log 150 \approx 2.176}$

Example 6: Solve the following equations.

a. $\log_7 7^{3x-5} = 14$

$\log_7 7^{(3x-5)} = 14$

$(3x - 5) = 14$ $\log_7 7$ are reverse operations.

$3x = 19$

$x = \dfrac{19}{3}$

b. $9^{3\log_3 x} = 8$

$9^{3\log_3 x} = 8$

$(3^2)^{3\log_3 x} = 8$

$3^{6\log_3 x} = 8$

$3^{\log_3 x^6} = 8$ $3^{\log_3}$ are reverse operations

$x^6 = 8$ $x = \sqrt[6]{8}$

Changing Base to Common Log Solve Logarithmic or Exponential Equations

1. If the logarithm is **not in base 10**, **convert it into an exponential form**.
2. **Common Log both sides of the equation**. Apply the law of logarithm and **bring the exponent inside the log out as a coefficient** and solve.

(Note: the log function of all scientific and graphing calculators are in base 10.)

$$a^x = y \qquad x = \frac{\log y}{\log a}$$

Example 7: Solve the following logarithmic equations. Round the answer to four decimal places.

a. $2^x = 25$

$2^x = 25$ (25 is **not** in family of base 2)

$\log 2^x = \log 25$ (Common log both sides)

$x(\log 2) = \log 25$

$x = \dfrac{\log 25}{\log 2}$

```
log(25)/log(2)
        4.64385619
2^Ans
                25
```

$x = 4.6439$

Note: We can verify by doing substituting the answer back into the original question.

b. $\log_3 50 = x$

$\log_3 50 = x$ (Convert to Exponential form)

$3^x = 50$ (50 is **not** in family of base 3)

$\log 3^x = \log 50$ (Common log both sides)

$x(\log 3) = \log 50$

$x = \dfrac{\log 50}{\log 3}$

```
log(50)/log(3)
       3.560876795
3^Ans
                50
```

$x = 3.5609$

Example 8: Simplify $(\log_4 8)(\log_8 12)$. Express as one single logarithm.

Let $\log_4 8 = x$ and $\log_8 12 = y$ (Convert each logarithm to Exponential form)

$4^x = 8$ and $8^y = 12$

$\log 4^x = \log 8$ and $\log 8^y = \log 12$ (Common log both sides)

$x(\log 4) = \log 8$ and $y(\log 8) = \log 12$

$x = \dfrac{\log 8}{\log 4}$ and $y = \dfrac{\log 12}{\log 8}$

$(\log_4 8)(\log_8 12) = (x)(y)$

$= \left(\dfrac{\cancel{\log 8}}{\log 4}\right)\left(\dfrac{\log 12}{\cancel{\log 8}}\right) = \dfrac{\log 12}{\log 4} \rightarrow 4^? = 12 \rightarrow \log_4 12$

5-3 Assignment: pg. 404−405 #1, 3, 7, 11, 15, 23, 27, 41, 43, 47, 49, 53, 63
Honours: #60

5-4 Exponential and Logarithm Equations

General Guidelines to Solve Exponential Equations:

1. **Recognize the numbers that belong to the same base family.** For example, (2, 4, 8, 16, 32, 64… belong to base two; 3, 9, 27, 81, 243… belong to base three). **Convert base sides to the same base**. **Equate the resulting exponents**.

2. **If the bases are *not* from the same family**, **Common Log Both Sides**, and solve algebraically. **Remember to view *log (number)* as *one item* when manipulating the equation**.

3. **For exponential equation that are quadratic like, look for common factor or use substitution to represent the exponential expression** like e^x or 2^x. **Solve** the resulting equation either **by factoring** or using **quadratic formula.**

Example 1: Find the solution of the exponential equation. Correct to four decimal places where needed.

a. $49^x = \left(\dfrac{1}{7}\right)^{3x-5}$

Both bases are in the base 7 family

$(7^2)^x = (7^{-1})^{(3x-5)}$ Power Rule $(a^m)^n = a^{m \cdot n}$

$7^{2x} = 7^{(-3x+5)}$ Equate exps (equal bases)

$2x = -3x + 5$

$5x = 5$

$x = 1$

b. $6^{3-x} = 2^{4x}$

Common Log both sides (6 and 2 are not in the same base family)

$\log 6^{(3-x)} = \log 2^{(4x)}$

$(3 - x)\log 6 = (4x)\log 2$ $\log_a x^y = y \log_a x$

$3\log 6 - x\log 6 = 4x\log 2$ Distribute

$3\log 6 = 4x\log 2 + x\log 6$ Collect like terms

$3\log 6 = x(4\log 2 + \log 6)$ Common factor (x)

$\dfrac{3\log 6}{(4\log 2 + \log 6)} = x$

```
3log(6)/(4log(2)
+log(6))
          1.17766616
```

$x \approx 1.1777$

c. $e^{5x-1} = 25$

Natural Log (ln) both sides to bring down exp

$\ln e^{5x-1} = \ln 25$

$(5x - 1)\ln e = \ln 25$ $\log_a x^y = y\log_a x$

$(5x - 1)(1) = \ln 25$ $\ln e = 1$

$5x = \ln(25) + 1$

$x = \dfrac{\ln(25)+1}{5}$

```
(ln(25)+1)/5
         .843775165
```

$x \approx 0.8438$

d. $350(1.02)^{8t} = 600$

$(1.02)^{8t} = \frac{600}{350}$ Divide both sides by 350

$(1.02)^{8t} = \frac{12}{7}$ Reduce

$\log(1.02)^{8t} = \log\left(\frac{12}{7}\right)$ Common log both sides

$8t\log(1.02) = \log\left(\frac{12}{7}\right)$ $\log_a x^y = y\log_a x$

$t = \dfrac{\log(12/7)}{8\log(1.02)}$

```
log(12/7)/(8log(
1.02))
         3.402304229
```

$t \approx 3.4023$

e. $x^2 3^x - 3^x = 0$

$3^x(x^2 - 1) = 0$ Common Factor $= 3^x$

$3^x(x + 1)(x - 1) = 0$ Factor Diff of Squares

Equate each factor to zero for solving x.

$3^x = 0$ $(x + 1) = 0$ $(x - 1) = 0$

No Solution $x = -1$ $x = 1$

$3^x > 0$ for any real number exponent

(Recall for the graph of $y = 3^x$, the range is $y > 0$)

f. $e^{2x} - 2e^x - 24 = 0$

$(e^x)^2 - 2e^x - 24 = 0$ Let $y = e^x$ and $y^2 = (e^x)^2$

$y^2 - 2y - 24 = 0$ Substitute

$(y - 6)(y + 4) = 0$ Factor Trinomial

Equate each factor to zero for solving y.

$(y - 6) = 0$ $(y + 4) = 0$

$y = 6$ $y = -4$

Substitute e^x back into y for each solution.

$e^x = 6$ $e^x = -4$ (No Solution)

$\ln e^x = \ln 6$ **$e^x > 0$ for any real number exponent**

$x(\ln e) = \ln 6$ (Recall for the graph of $y = e^x$, the range is $y > 0$)

$x = \ln 6$

General Guidelines to Solve Logarithmic Equations:

1. If the **base is the same on all logarithmic terms**, **combine them into a single logarithm on either side**.
2. **Convert the single logarithmic equation into an exponential equation** and solve.
3. **Verify by substituting x back into the equation** so that for each logarithmic term, $\log_a (x + b)$, the expression inside the log term, $(x + b) > 0$. **Disregarded** all other solutions as **Extraneous Solutions**.

Example 2: Solve the logarithmic equations below.

a. $\ln(5x - 2) = 3$

Can't do much in log form

$\log_e (5x - 2) = 3$

$e^3 = (5x - 2)$ Change to exp form

$e^3 + 2 = 5x$ Isolate x to solve

$\dfrac{e^3 + 2}{5} = x$

b. $2 \log_5 x = \log_5 2 + \log_5 (x + 12)$

Combine each side to a single log term

$\log_5 x^2 = \log_5 [2 (x + 12)]$

$x^2 = 2 (x + 12)$ $\log_5$ cancel on both sides

$x^2 = 2x + 24$

$x^2 - 2x - 24 = 0$ Solve quadratic by bring all terms to one side and factor

$(x - 6)(x + 4) = 0$

$(x - 6) = 0$ $(x + 4) = 0$

$x = 6$ ✓ $\cancel{x = -4}$

Verify: $\log_5 x = \log_5 6$ ✓ $\log_5 (x + 12) = \log_5 (6 + 12)$ ✓

Verify: $\log_5 x = \log_5 -4$ ✗ $\therefore$ $x = -4$ **is extraneous**

c. $\log_3 (x^2 - 3x + 5) = 2$

Can't do much in log form; change to exp form

$(x^2 - 3x + 5) = 3^2$

$x^2 - 3x + 5 = 9$

$x^2 - 3x - 4 = 0$ Quadratic (Equate to 0 and Factor)

$(x - 4)(x + 1) = 0$

$(x - 4) = 0$ $(x + 1) = 0$

$x = 4$ ✓ $x = -1$ ✓

Verify: Substitute soln's into $\log_3 (x^2 - 3x + 5)$

$\log_3 ((4)^2 - 3(4) + 5) = \log_3 9$ ✓

$\log_3 ((-1)^2 - 3(-1) + 5) = \log_3 9$ ✓

d. $\log_2 (2x + 4) - \log_2 (x - 1) = 3$

$\log_2 \left[\dfrac{(2x+4)}{(x-1)}\right] = 3$ Combine left side to a single log term

$\left[\dfrac{(2x+4)}{(x-1)}\right] = 2^3$ Can't do much in log form; change to exp form

$\dfrac{(2x+4)}{(x-1)} = 8$ Cross Multiply

$(2x + 4) = 8(x - 1)$

$2x + 4 = 8x - 8$

$12 = 6x$

$x = 2$ ✓

Verify:

$\log_2 (2x + 4) = \log_2 (2(2) + 4)$ ✓

$\log_2 (x - 1) = \log_2 ((2) - 1)$ ✓

e. $\log_9 (x + 3) = 1 - \log_9 (x - 5)$

$\log_9 (x + 3) + \log_9 (x - 5) = 1$ Bring all log terms to one side and combine

$\log_9 [(x + 3)(x - 5)] = 1$

$(x + 3)(x - 5) = 9^1$ Change to exp form

$x^2 - 2x - 15 = 9$ Quadratic (Equate to 0 and Factor)

$x^2 - 2x - 24 = 0$

$(x - 6)(x + 4) = 0$

$(x - 6) = 0$ $(x + 4) = 0$

$x = 6$ ✓ $\cancel{x = -4}$

Verify:

$\log_9 (x + 3) = \log_9 (6 + 3)$ ✓

$\log_9 (x - 5) = \log_9 (6 - 5)$ ✓

Verify: $\log_9 (x + 3) = \log_9 (-4 + 3) = \log_9 -1$ ✗

$\therefore$ $x = -4$ **is extraneous**

f. $\log_4 (\log (\ln x)) = 2$

$\log_4 (\log (\ln x)) = 2$ Work from the outside, change each log form to exp form

$\log (\ln x) = 4^2$

$\log_{10} (\ln x) = 16$

$\ln x = 10^{16}$

$\log_e x = (10^{16})$

$x = e^{(10^{16})}$

Example 3: Solve the inequality, $\log_2 (x - 4) + \log_2 (10 - x) > 3$.

$\log_2 (x - 4) + \log_2 (10 - x) > 3$ — Bring all log terms to one side and combine

$\log_2 [(x - 4)(10 - x)] > 3$

$(x - 4)(10 - x) > 2^3$ — Change to exp form

$10x - x^2 - 40 + 4x > 8$

$-x^2 + 14x - 40 > 8$ — Quadratic (Equate to 0)

$-x^2 + 14x - 48 > 0$ — Divide both sides by -1 (switch direction of inequality sign)

$x^2 - 14x + 48 \boxed{<} 0$

$(x - 6)(x - 8) < 0$

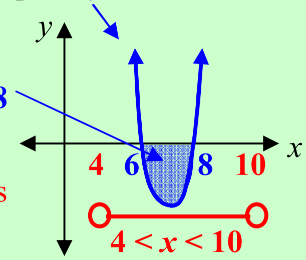
$(x - 6) = 0$ $\quad$ $(x - 8) = 0$ — Factor to find solutions

$\boldsymbol{x = 6}$ $\quad$ $\boldsymbol{x = 8}$

Using what we know from graphing polynomial, the graph of $\boldsymbol{y = (x - 6)(x - 8)}$ would be,

Note that $y < 0$ when $6 < x < 8$

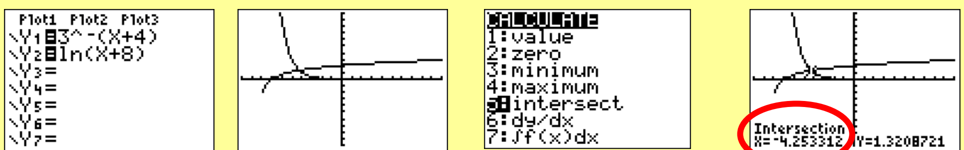
The domain of the original equation is $(x - 4) > 0$ and $(10 - x) > 0$. Hence, $x > 4$ and $x < 10$



Since the solution, $\boldsymbol{6 < x < 8}$, **is within the domain**, $\boldsymbol{4 < x < 10}$, **it is a valid solution.** $\boldsymbol{6 < x < 8}$

Example 4: Using a graphing calculator, solve $3^{-(x + 4)} = \ln (x + 8)$ to the nearest thousandth.

Y= $\quad$ **GRAPH** $\quad$ **2nd** **CALC** / **TRACE** $\quad$ Press **ENTER** three times



$\boldsymbol{x \approx -4.253}$

Example 5: Sarah invested \$4000 into a term deposit account that pays 6.6%/a compound monthly. To the nearest tenth, how many years would it take for her investment to be double in value?

$P = \$4000$

$A = \$8000$ (double P)

$r = 0.066$

$n = 12$ (compound monthly)

$\boldsymbol{t = ?}$

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

$$8000 = 4000\left(1 + \tfrac{0.066}{12}\right)^{12t}$$

$$\tfrac{8000}{4000} = (1 + 0.0055)^{12t}$$

$$2 = (1.0055)^{12t}$$

$$\log 2 = \log (1.0055)^{12t}$$

$$\log 2 = 12t \log (1.0055)$$

$$\frac{\log 2}{12\log(1.0055)} = t$$

```
log(2)/(12log(1.
0055)
      10.53108474
```

$\boldsymbol{t \approx 10.5}$ **yr**

<u>Exponential Problem that Compound Continuously</u>:

For exponential problem that increases or decreases continuously, that is, n is extremely big, the base of the exponential equation reduces to the natural exponential function, e.

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt} \xrightarrow{n \to \infty} A(t) = A_0 e^{rt}$$

$A(t)$ = Final Amount after t years

A_0 = Initial Amount

r = Rate of Increase ($+r$) / Decrease ($-r$) per year

Example 6: The population of a small town was 50,000 people in the year 2000. At an annual growth rate of 4.25%, in what year will the population reach 120,000 people?

$A_0 = 50{,}000$

$A(t) = 120{,}000$

$r = 0.0425$

$t = ?$

$$A(t) = A_0 e^{rt}$$

$$120{,}000 = 50{,}000\, e^{0.0425t}$$

$$\tfrac{120{,}000}{50{,}000} = e^{0.0425t}$$

$$2.4 = e^{0.0425t}$$

$$\ln 2.4 = \ln e^{0.0425t}$$

$$\ln 2.4 = 0.0425t \ln e \quad (\ln e = 1)$$

$$\frac{\ln 2.4}{0.0425} = t$$

```
ln(2.4)/0.0425
      20.59926441
```

$\boldsymbol{t \approx 20.6}$ **after 2000**

In the year 2020, the population will reach 120,000 people

5-4 Assignment: pg. 414−415 #5, 9, 13, 19, 25, 27, 31, 37, 43, 47, 55, 59, 63, 67, 71, 76
Honours: #54, 65, 77

5-5 Modeling with Exponential and Logarithm Functions

Applications with Exponential Functions

In nature, many scenarios involving continuous exponential growth and decay can be summarized in the following function. The notation of the variable might be different, but essentially it is from this main natural exponential function.

$$A(t) = A_o e^{rt}$$

$A(t)$ = **Final Amount after t years**
A_o = **Initial Amount**
r = **Rate of Increase (+r) / Decrease (−r) per year**

Population Growth: - a population of people, animals, bacteria follows the continuous exponential growth model. Assuming a steady growth rate over a period of time, we can use the following function. (Note the variable A is replaced by N.)

$$N(t) = N_o e^{rt}$$

$N(t)$ = **Final Population after t years, hours, minutes, or seconds**
N_o = **Initial Population**
r = **Rate of Increase per year, hour, minute, or second**

Example 1: E. coli bacteria replicates quite quickly. It can double its population in 20 minutes. Determine the growth rate of E. coli to the nearest fourth decimal place.

$\frac{N(t)}{N_o} = 2$ (doubles)

$t = 20$ mins $\quad r = ?$

$N(t) = N_o e^{rt}$

$\frac{N(t)}{N_o} = e^{rt}$

$2 = e^{r(20)}$

$\ln 2 = \ln e^{20r}$

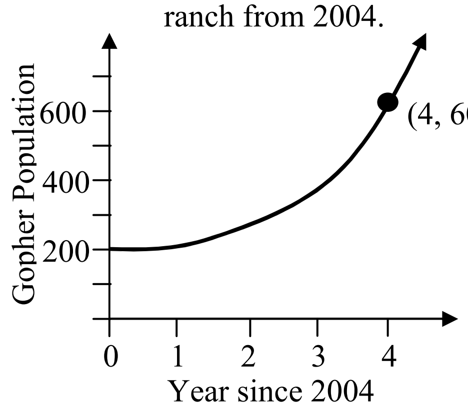
$\ln 2 = 20r \ln e \quad (\ln e = 1)$

$\frac{\ln 2}{20} = r$

```
ln(2)/20
        .034657359
```

$r \approx 0.0347$

Example 2: The following graph shows the prairie dogs (sometimes refer to as gophers) population on a ranch from 2004.



a. What was the gopher population in 2004?

b. Find the function that models the graph.

c. In what year will the prairie dog population grow to 1000?

a. In the year 2004, we look at the y-intercept because the x-axis is defined as year since 2004.

(0, 200) → there are 200 gophers in 2004.

b. From the point, (4, 600), of the graph, $t = 4$ and $N(t) = 600$. We also know that the initial population, $N_o = 200$. Solving for r, we have

$N(t) = N_o e^{rt}$

$600 = 200e^{r(4)}$

$\frac{600}{200} = e^{4r}$

$3 = e^{4r}$

$\ln 3 = \ln e^{4r}$

$\ln 3 = 4r \ln e \quad (\ln e = 1)$

$\frac{\ln 3}{4} = r$

$r \approx 0.2647$

```
ln(3)/4
        .2746530722
ln(5)/0.2747
        5.858893019
```

$N(t) = 200e^{0.2647t}$

c. $t = ?$ when $N(t) = 1000$

$N(t) = 200e^{0.2647t}$

$1000 = 200e^{0.2647t}$

$\frac{1000}{200} = e^{0.2647t}$

$5 = e^{0.2647t}$

$\ln 5 = \ln e^{0.2647t}$

$\ln 5 = 0.2647t \ln e$

$(\ln e = 1)$

$\frac{\ln 5}{0.2647} = t$

$t \approx 5.8589$ yrs
In the year 2009

Radioactive Decay: - the exponential decrease of the amount of radioactive material of a radioactive isoptope (atoms of an element with different number of neutrons than the most common kind). (Note the variable A is replaced by m.)

Half-Life (h): - the time it takes to decrease the mass of a radioactivie material by half compared to the original mass.

$$m(t) = m_0 e^{-rt} \qquad r = \frac{\ln 2}{h}$$

$m(t)$ = Final Mass after t years, hours, minutes, or seconds
m_0 = Initial Mass
r = Rate of Decrease per year, hour, minute, or second
h = Half-life in year, hour, minute, or second

Example 3: Iodine-135 is commonly use as a radiotracer to detect problems in the thyroid gland. It has a half-life of 8 days. How long does it take a dosage of 0.9 mg of iodine-135 to decay to 0.00024 mg (the sensitivity level of most radioactivity detector used by law enforcement agencies)?

h = 8 days
r = ?

m_0 = 0.9 mg
$m(t)$ = 0.00024 mg
t = ?

We have to find r first in order to write the equation. Then, we can calculate t for the specified amounts.

$$r = \frac{\ln 2}{h} = \frac{\ln 2}{8} \qquad r \approx 0.086643$$

(Since we need r for subsequent calculations, we should keep as many decimal places as possible.)

$$m(t) = m_0 e^{-rt}$$
$$0.00024 = 0.9e^{-0.086643t}$$
$$\frac{0.00024}{0.9} = e^{-0.086643t}$$
$$\ln\left(\tfrac{0.00024}{0.9}\right) = \ln e^{-0.086643t}$$
$$\ln\left(\tfrac{0.00024}{0.9}\right) = -0.086643t \ln e \qquad (\ln e = 1)$$
$$\frac{\ln\left(\tfrac{0.00024}{0.9}\right)}{-0.086643} = t \qquad t \approx 95 \text{ days}$$

```
ln(2)/8
          .0866433976
ln(0.00024/0.9)/
-0.086643
          94.98183487
```

Newton's Law of Cooling: - the exponential decrease of an object's warmer temperature to the cooler surrounding. (Note the variable A is replaced by T.)

$$T(t) = T_s + D_0 e^{-kt}$$

$T(t)$ = Final Temperature after t amount of time
T_s = Temperature of the Surrounding
D_0 = Difference between Temperature of the Warm Object and the Surrounding
k = Rate of Cooling unique to the type of object

Example 4: A cup of hot coffee at 85°C was in a closed thermos mug. An hour later, the temperature of the coffee was 72°C. The room temperature was kept at a constant temperature of 20°C.

a. What was the rate of cooling of this thermos?
b. How long does the coffee to cool to 37°C?

a.

T_0 = 85°C
T_s = 20°C
$D_0 = T_0 - T_s$
= 85°C − 20°C
D_0 = 65°C

$T(t)$ = 72°C
t = 1 hour $\qquad r$ = ?

$$T(t) = T_s + D_0 e^{-kt}$$
$$72 = 20 + 65e^{-k(1)}$$
$$72 - 20 = 65e^{-k}$$
$$52 = 65e^{-k}$$
$$\frac{52}{65} = e^{-k}$$
$$0.8 = e^{-k}$$
$$\ln 0.8 = \ln e^{-k}$$
$$\ln 0.8 = -k \ln e \qquad (\ln e = 1)$$
$$-\ln 0.8 = k \qquad k \approx 0.2231436$$

```
-ln(0.8)
          .2231435513
```

b. Find t when $T(t)$ = 37°C

$$T(t) = T_s + D_0 e^{-kt}$$
$$37 = 20 + 65e^{-0.2231436t}$$
$$37 - 20 = 65e^{-0.2231436t}$$
$$17 = 65e^{-0.2231436t}$$
$$\tfrac{17}{65} = e^{-0.2231436t}$$
$$\ln\left(\tfrac{17}{65}\right) = -0.2231436t \ln e \qquad (\ln e = 1)$$
$$\frac{\ln\left(\tfrac{17}{65}\right)}{-0.2231436} = t \qquad t \approx 6 \text{ hours}$$

```
ln(17/65)/-0.223
1436
          6.010362501
```


Applications with Logarithmic Functions

Certain physical quantities such as chemical concentrations and vibrational intensities can occur over a very large range. In these cases, such quantities can be easily rewritten after they have gone through a logarithmic function. The notation of the variable might be different, but essentially it is from this main common logarithmic function.

$f(x) = \log x$	$f(x)$ = Logarithmic Scale of the physical quantity (x)

pH Scale: - a logarithmic scale to measure the acidity of a chemical by using the **hydrogen ion concentration, $[H^+]$** (measures in mol/L – an amount of particle per unit volume) of a solution.
- **an increase of pH of 1 is equivalent to a decrease of $[H^+]$ by a factor of 10.**

pH Scale

0	Acidic	7 Neutral	Basic	14

Example: pH of Some Common Substances

Substance	pH
1 M of HCl	0.00
Stomach Acid	2.00
Lemon Juice	2.50
Vinegar	3.00

Substance	pH
Milk	6.30
Rain Water	6.70
Pure Water	7.00
Blood	7.50

Substance	pH
Sea Water	8.20
Milk of Magnesia	10.50
Ammonia as Household Cleaner	12.00
1 M of NaOH	14.00

$\text{pH} = -\log [H^+]$	$[H^+]$ = hydrogen ion concentration in (mol/L)

Example 5: A properly diluted commercial bleach solution has a $[H^+]$ of 5.92×10^{-5} mol/L.

a. What is the pH of this diluted bleach solution?

b. A competitor uses another acid solution as a whitener. The pH of this product is 5.45. How many times stronger is the $[H^+]$ of the commercial bleach solution compared to this whitener?

a. $[H^+] = 5.92 \times 10^{-5}$ mol/L, **pH = ?**

$\text{pH} = -\log [H^+]$

$\text{pH} = -\log (5.92 \times 10^{-5})$

To enter scientific notations, use **2nd** **EE ,**

```
-log(5.92E-5)
        4.227678293
```

pH = 4.22

b. $\text{pH}_{\text{whitener}} = 5.45$, $[H^+]_{\text{whitener}} = ?$, $\dfrac{[H^+]_{\text{bleach}}}{[H^+]_{\text{whitener}}} = ?$

$\text{pH} = -\log [H^+]$

$-\text{pH} = \log [H^+]$

$10^{-\text{pH}} = [H^+]$

$[H^+]_{\text{whitener}} = 10^{-(5.45)}$

$[H^+]_{\text{whitener}} = 3.548 \times 10^{-6}$ mol/L

```
10^-5.45
        3.548133892E-6
5.92E-5/Ans
        16.68482695
```

$\dfrac{[H^+]_{\text{bleach}}}{[H^+]_{\text{whitener}}} = \dfrac{5.92 \times 10^{-5}}{3.548... \times 10^{-6}}$

$\dfrac{[H^+]_{\text{bleach}}}{[H^+]_{\text{whitener}}} = 16.7$ **times**

Richter Scale: - a logarithmic scale to measure the intensity of an earthquake based on the **intensity of a "standard" earthquake (S)** with a vibrational amplitude of 1 micron (10^{-6} m = 1 μm) on a seismograph 1 km from the epicentre.
- **an increase of Richter Magnitude of 1 is equivalent to an increase of earthquake intensity by a factor of 10.**

$M = \log \left(\dfrac{I}{S}\right)$	M = Richter Scale Magnitude I = Intensity of the Measured Earthquake S = Intensity of the Standard Earthquake

Example 6: A day after Christmas of 2004, an underwater earthquake near Sumatra, Indonesia started a tsunami that killed over 227,000 people. It measured with a magnitude of 9.1 on the Richter scale. How many times more intense is this earthquake compared to the earthquake in San Francisco of 1906 (approximately 3000 fatalities), which measured 7.8 on the Richter scale?

$M_{\text{Indo}} = 9.1$
$M_{\text{SF}} = 7.8$

$\dfrac{I_{\text{Indo}}}{I_{\text{SF}}} = ?$

Starting with the formula, we first isolate I.

$$M = \log\left(\frac{I}{S}\right)$$

$$10^M = \frac{I}{S} \quad \text{(Change to Exp form)}$$

$$(10^M)(S) = I$$

Now, substitute the manipulated formula into the ratio.

$$\frac{I_{\text{Indo}}}{I_{\text{SF}}} = \frac{\left(10^{M_{\text{Indo}}}\right)\cancel{(S)}}{\left(10^{M_{\text{SF}}}\right)\cancel{(S)}}$$

$$\frac{I_{\text{Indo}}}{I_{\text{SF}}} = \frac{\left(10^{M_{\text{Indo}}}\right)}{\left(10^{M_{\text{SF}}}\right)} = \frac{\left(10^{9.1}\right)}{\left(10^{7.8}\right)} = 10^{(9.1-7.8)} = 10^{1.3}$$

```
10^1.3
          19.95262315
```

$$\frac{I_{\text{Indo}}}{I_{\text{SF}}} = 19.9526\ldots \qquad \boldsymbol{\frac{I_{\text{Indo}}}{I_{\text{SF}}} \approx 20 \text{ times}}$$

<u>Decibel Scale</u> (dB): - a logarithmic scale to measure the intensity or loudness of sound based on a **refernece intensity (I_o)** of 10^{-12} W/m^2 (Watts per square metre – power rating over an unit area).
- **an increase of Decibel Level of 10 is equivalent to an increase of sound intensity by a factor of 10.**

$$\boldsymbol{B = 10 \log\left(\frac{I}{I_o}\right)}$$

B = Loudness in Decibels
I = Intensity of the Measured Sound (W/m^2)
I_0 = Reference Intensity (10^{-12} W/m^2)

Example 7: A heavy metal rock concert can reach a sound intensity of 15 W/m^2 near the front of the stage. What is the equivalent decibel level?

$I = 15$ W/m^2
$I_0 = 10^{-12}$ W/m^2

$B = ?$

$$B = 10 \log\left(\frac{I}{I_o}\right)$$

$$B = 10 \log\left(\frac{15}{10^{-12}}\right)$$

```
10log(15/10^-12)
         131.7609126
```

$$\boldsymbol{B = 131.8 \text{ dB}}$$

5-5 Assignment: pg. 427−430 #1, 5, 7, 12, 17, 23, 27, 33, 36; Honours: #19 and 41

