ccuracy more reliable? A problem such as generating the digit
ovides a proving ground for technological improvement.
vertheless, probably the most honest explanation of such [illegible]r-
e is simple human curiosity about the unknown. Virtually any
m without an easy solution will lure at least a few people to pur-
sometimes obsessively. The history of both the progress and the
f the human race is dotted with the achievements and the misad-
es of such people. Not knowing in advance which questions will
hich way adds a risk factor that makes them more inviting. In
matics, as in any sport, overcoming the challenge of the untried
e unknown is its own reward.

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77818577805 32[illegible]12268066130019278766111[illegible]5909216420198
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The first 1000 decimal places of π

Closer Look: Beckmann's [11] is a readable book about the
y of π. Also worth looking at is [16], which collects many arti-
ncluding some of the original sources (for example, it contains a
ing from Shanks's original publication). The latest information
Prof. Kanada's computations can be found via the home page of
Kanada's laboratory at pi2.cc.u-tokyo.ac.jp.

8 The Cossic Art

Writing Algebra with Symbols

When you think of *algebra*, what comes to mind first? Do you think of equations or formulas made up of x's and y's and other letters, strung together with numbers and arithmetic symbols? Many people do. In fact, many people regard algebra simply as a collection of rules for manipulating symbols that have something to do with numbers.

There's some truth in that. But describing algebra solely in terms of its symbols is like describing a car by its paint job and body style. What you see is *not* all you get. In fact, like a car, most of what makes algebra run is "under the hood" of its symbolic appearance. Nevertheless, just as an automobile's body styling can affect its performance and value, so does the symbolic representation of algebra affect its power and usefulness.

An algebra problem, regardless of how it's written, is a question about numerical operations and relations in which an unknown quantity must be deduced from known ones. Here's a simple example:

> Twice the square of a thing is equal to five more than three times the thing. What is the thing?

Despite the absence of symbols, this is clearly an algebra question. Moreover, the word "thing" was a respectable algebraic term for a very long time. In the 9th century, Al-Khwārizmī (whose book title, *al-jabr w'al muqābala*, is the source of the word "algebra") used the word *shai* to mean an unknown quantity. When his books were translated into Latin, this word became *res*, which means "thing". For instance, John of Seville's 12th-century elaboration of Al-Khwārizmī's arithmetic contains this question, which begins "Quaeritur ergo, quae res...":[1]

> It is asked, therefore, what thing together with 10 of its roots or what is the same, ten times the root obtained from it, yields 39.

In modern notation, this would be written either as $x + 10\sqrt{x} = 39$ or as $x^2 + 10x = 39$. (An "X" appears in the Latin version of this question, but it's actually the Roman numeral for 10. To avoid such

[1] See p. 336 of [23] for both the original Latin and this translation.

confusions and emphasize more significant variations in notation, we use familiar numerals in all these algebra examples.[2])

Some Latin texts used *causa* for Al-Khwārizmī's *shai*, and, when these books were translated into Italian, *causa* became *cosa*. As other mathematicians studied these Latin and Italian texts, the word for the unknown became *Coss* in German. The English picked up on this and called the study of questions involving unknown numbers "the Cossic Art" (or "Cossike Arte" in the spelling of those days) — literally, "the Art of Things".

Like most of our familiar algebraic symbols, the x and other letters we now use to represent unknown numbers are relative newcomers to the "art." Many early algebraic symbols were just abbreviations for frequently used words: p or $\tilde{p}$ or $\bar{p}$ for "plus," m or $\tilde{m}$ or $\bar{m}$ for "minus," and so on. While they saved writing time and print space, they did little to promote a deeper understanding of the ideas they expressed. Without consistent and illuminating symbolism, algebra was indeed an art, an often idiosyncratic activity that depended heavily on the skill of its individual practitioners. Just as standardization of parts was a critical step in the mass production of Henry Ford's automobiles, so the standardization of notation was a critical step in the use and progress of algebra.

Good mathematical notation is far more than efficient shorthand. Ideally, it should be a universal language that clarifies ideas, reveals patterns, and suggests generalizations. If we invent a really good notation, it sometimes seems to think for us: just manipulating the notation achieves results. As Howard Eves once said, "A formal manipulator in mathematics often experiences the discomforting feeling that his pencil surpasses him in intelligence."[3]

Our current algebraic notation is close to this ideal, but its development has been long, slow, and sometimes convoluted. For a flavor of that development, we'll look at various ways in which a typical algebraic equation would likely have been written in different times and places during the progress of algebra in Europe. (To highlight the notational development, we use English in place of Latin or other languages when words, rather than symbols, would be used.)

Here is an equation containing some common ingredients of early algebraic investigations:

$$x^3 - 5x^2 + 7x = \sqrt{x+6}$$

[2]See Sketch 1 for an account of how numerals have changed over the years.

[3]See [45], entry 251.

In 1202, Leonardo of Pisa would have written that equation (perhaps rearranged for clarity) entirely in words, something like this:

> The cube and seven things less five squares is equal to the root of six more than the thing.

This approach to writing mathematics is usually called *rhetorical*, in contrast to the symbolic style we use today. In the 13th and 14th centuries, European mathematics was almost entirely rhetorical, with occasional abbreviations here and there. For instance, Leonardo began to use R for "square root" in some of his later writings.

Late in the 15th century, some mathematicians started to use symbolic expressions in their work. Luca Pacioli, whose *Summa de Arithmetica* of 1494 served as a main source of Europe's introduction to the cossic art, would have written

cu.m̃.5.ce.p̃.7.co.——℞v.co.p̃.6.

In this notation, *co* is an abbreviation for "cosa," the unknown quantity. The abbreviations *ce* and *cu* are for "censo" and "cubo," words that the Italian mathematicians used for the square and the cube of the unknown, respectively. Notice that we refer to *the* unknown here. A fundamental weakness of this notation was its inability to represent more than one unknown in an expression. (By way of contrast, the Hindus had been using the names of colors to represent multiple unknowns as early as the 7th century.) Some other interesting features of Pacioli's notation are the dots that separate each item from the next, a long dash for equality, and the symbol ℞ to denote square root. The grouping of terms after the root sign was signaled by v, an abbreviation for "universale." The notation used in Girolamo Cardano's *Ars Magna* half a century later in Italy was almost identical to this.

In early 16th-century Germany, some of the symbols we use now began to appear. The $+$ and $-$ signs were adopted from commercial arithmetic and the "surd" symbol, $\surd$, for square root evolved, some say from a dot with a "tail," others say from a handwritten r. Equality was noted by abbreviating either the Latin or German word for it, and the grouping of terms (such as the sum after the $\surd$ sign) was signaled by dots. Thus, in Christoff Rudolff's *Coss* of 1525 (which has an impossibly long formal title) or Michael Stifel's *Arithmetica Integra* of 1544, our equation might have appeared as

ȼe − 5ʓ + 7ℛ *aequ.* √.ℛ + 6.

As in the earlier Italian notation described above, different powers of the unknown had distinct, unrelated symbols. Its first power was called the root (*radix*) and represented by $\mathfrak{R}$. The symbol for its square was $\mathfrak{z}$, a small script z which was the first letter of its German name, *zensus*. The third power, *cubus*, was symbolized by $\mathcal{C}$. Higher powers of the unknown were written by combining the square and cube symbols multiplicatively, when possible; the fourth power was $\mathfrak{z}\mathfrak{z}$, the sixth power was $\mathfrak{z}\mathcal{C}$, and so on. Higher prime powers were handled by introducing new symbols.

Easier ways to denote powers of the unknown had already begun to emerge in other countries. One of the most creative instances of this appeared in a 1484 manuscript by Nicholas Chuquet, a French physician. Like others of his time, Chuquet confined his attention to powers of a single unknown. However, he denoted the successive powers of the unknown by putting numerical superscripts on the coefficients. For example, to denote $5x^4$ he would write 5^4. He did a similar thing for roots, writing $\sqrt[3]{5}$ as $\mathcal{R}^3.5$. Chuquet was also well ahead of his time in treating zero as a number (particularly as an exponent) and in using an underline for aggregation. If our example equation had appeared in his manuscript, it would have looked like this:

$$1^3.\bar{m}.5^2.\bar{p}.7^1.\ \textit{montent}\ \mathcal{R}^2.\underline{1^1.\bar{p}.6^0}.$$

Unfortunately for the development of algebraic notation, Chuquet's work was not published at the time it was written, so his innovative ideas were known only to a few mathematicians by the beginning of the 16th century. This system of denoting powers of the unknown reappeared in 1572 in the work of Rafael Bombelli, who placed the exponents in small cups above the coefficients. Bombelli's work was more widely known than that of Chuquet, but his notation was not immediately adopted by his contemporaries. In the 1580s it was picked up by Simon Stevin of Belgium, a military engineer and inventor, who used circles around the exponents. Stevin's mathematical writing emphasized the convenience of decimal arithmetic. Some of his publications were translated into English early in the 17th century, thereby carrying both his ideas and his notation across the English Channel.

A major breakthrough in notational flexibility and generality was made by François Viète in the last decade of the 16th century. Viète was a lawyer, a mathematician, and an advisor to King Henri IV of France with duties that included deciphering messages written in secret codes. His mathematical writings focused on methods of solving algebraic equations, and to clarify and generalize his work he introduced a revolutionary notational device. In Viète's own words:

> In order that this work may be assisted by some art, let the given magnitudes be distinguished from the undetermined

> unknowns by a constant, everlasting and very clear symbol, as, for instance, by designating the unknown magnitude by means of the letter A or some other vowel... and the given magnitudes by means of the letters B, G, D or other consonants.[4]

Using letters for both constants and unknowns allowed Viète to write general forms of equations, instead of relying on specific examples in which the particular numbers chosen might improperly affect the solution process. Some earlier writers had experimented with using letters, but Viète was the first to use them as an integral part of algebra. It may well be that the emergence of this powerful notational device was delayed because the Hindu-Arabic numerals were not commonly used until well into the 16th century. Prior to that, Roman numerals (and Greek numerals before them) were used for writing numbers, and these systems used letters of the alphabet for specific quantities.

As soon as equations contained more than one unknown, it became clear that the old exponential notation was insufficient. It would not do to write $5^3 + 7^2$ if one meant $5A^3 + 7E^2$. In the 17th century, several competing notational devices for this appeared almost simultaneously. In the 1620s, Thomas Harriot in England would have written it as $5aaa + 7ee$. In 1634, Pierre Hérigone of France wrote unknowns with coefficients before and exponents after, as in $5a3 + 7e2$. In 1636, James Hume (a Scotsman living in Paris) published an edition of Viète's algebra with exponents elevated and in small Roman numerals, as in $5a^{iii} + 7e^{ii}$. In 1637, a similar notation appeared in René Descartes's *La Géométrie*, but with the exponents written as small Hindu-Arabic numerals, as in $5a^3 + 7e^2$. Of these notations, Harriot's and Hérigone's were the easiest to typeset, but conceptual clarity won out over typographical convenience and Descartes's method eventually became the standard notation used today.

Descartes's influential work is also the source of some other notational devices that have become standard. He used lowercase letters from the end of the alphabet for unknowns and lowercase letters from the beginning of the alphabet for constants. He also used an overline bar from the $\surd$ sign to indicate its scope. However, he introduced the symbol $\propto$ for equality. Thus, Descartes's version of our sample equation would be very much, but not entirely, like our own:

$$x^3 \text{ -- } 5xx + 7x \propto \sqrt{x+6}$$

[4]From Viète's *In artem analyticam Isagoge* of 1591, as translated by J. Winfree Smith. See [84], p 340.

The $=$ sign for equality, proposed in 1557 by Robert Recorde[5] and widely used in England, was not yet popular in continental Europe. In the 17th century it was only one of several different ways of symbolizing equality, including $\sim$ and the ∞ sign of Descartes. Moreover, $=$ was being used to denote other ideas at this time, including parallelism, difference, and "plus or minus." Its eventual universal acceptance as the symbol for "equals" is probably due in large part to its adoption by both Isaac Newton and Gottfried Leibniz. Their systems of the calculus dominated the mathematics of the late 17th and early 18th centuries, so their notational choices became widely known. During the 18th century, the superior calculus notation of Leibniz gradually superseded that of Newton. Had Leibniz chosen to use Descartes's symbol instead of Recorde's, we might be using ∞ for equality today.

This sketch has tried to capture the flavor of the long, erratic, sometimes perverse way in which algebraic symbolism has developed. In hindsight, "good" notational choices have proved to be powerful stimuli for mathematical progress. Nevertheless, those choices often were made with little awareness of their importance at the time. The evolution of exponential notation is a prime example of this. Powers of an unknown quantity were trapped for centuries by the limited geometric intuition of squares and cubes, and the notation reinforced this confinement. Descartes finally liberated them by treating squares, cubes, and the like as magnitudes independent of geometric dimension, giving a new legitimacy to x^4, x^5, x^6, and so on. From there the notation itself suggested natural extensions — to negative integral exponents (reciprocals), to rational exponents (roots of powers), to irrational exponents (limits of roots of powers), and even to complex exponents. And in the 20th century, this exponential notation was reconnected with the geometric idea of dimension to help lay the foundation of a new field of mathematical investigation: fractal geometry.

For a Closer Look: There are treatments of the evolution of algebraic notation in most surveys of the history of mathematics. For specific information on the history of mathematical notations, the best reference is still [23], though *Earliest Uses of Various Mathematical Symbols*, a web site at http://members.aol.com/jeff570/mathsym.html maintained by Jeff Miller, is now a serious contender. For more on the history of algebra, see [10] and [138].

[5] See Sketch 2 for more details about this.

9 Linear Thinking

Solving First Degree Equations

Problems that reduce to solving an equation of degree one arise naturally whenever we apply mathematics to the real world. It's not surprising, then, to find that almost everyone who studied mathematics, from the Egyptian scribes to the Chinese civil servants, developed techniques for solving such problems.

The Rhind Papyrus, a collection of problems probably used for training young scribes in Ancient Egypt, contains several problems of this kind. Some are simple and straightforward, others quite complicated. Here's a simple one:

A quantity; its half and its third are added to it. It becomes 10.

In our notation, that is just the equation

$$x + \frac{1}{2}x + \frac{1}{3}x = 10.$$

(Keep in mind, though, that this kind of symbolism was still far in the future, as explained in Sketch 8.) The scribe is instructed to solve it just as we would: divide 10 by $1 + \frac{1}{2} + \frac{1}{3}$.

Quite often, however, the problems in the Rhind Papyrus are solved by a very different method.

A quantity; its fourth is added to it. It becomes 15.

Instead of dividing 15 by $1\frac{1}{4}$, the scribe proceeds as follows. He assumes (or *posits*) that the quantity is 4. (Why 4? Because it's easy to compute a fourth of 4.) If you take 4 and add its fourth to it, you get $4+1=5$. So we wanted 15, but we got 5; we need to multiply what we got (that is, 5) by 3 to get what we wanted to get (that is, 15). So we take our guess and multiply it by 3. Our guess was 4, so the answer is $3\times4=12$.

This method is known as *false position*: we posit an answer that we don't really expect to be the right one, but which makes the computations easy. Then we use the incorrect result of that guess to find the number by which we need to multiply our guess to get the correct answer.

Symbols make this easy to understand. The equation we're solving looks like $Ax = B$. If we multiply x by a factor, so that it becomes kx, we see that

$$A(kx) = k(Ax) = kB.$$

So scaling the input by some factor scales the output by the same factor. This is what allows the method of false position to work; we use our guess to find the right factor.

Throughout antiquity, the method of false position was used to solve linear equations, including some pretty complicated ones. These range all the way from practical problems to more fanciful problems with a recreational flavor.

However, this method can only be applied to equations of the form $Ax = B$. If, instead, the equation were $Ax + C = B$, then it is no longer true that multiplying x by a factor causes B to change by the same factor, and this simple version of the method breaks down. We might try subtracting C from both sides, but that isn't always as easy as it sounds, because the expression on the left side might initially be very complicated, and finding the correct constant to subtract would require us to simplify it to the form $Ax + C$.

Instead, a way was found to extend the basic idea to equations of that type without any such algebraic manipulation. It is called the method of *double false position*. This is such an effective method for solving linear equations that it continued to be used long after the invention of algebraic notations. In fact, since it doesn't require any algebra, it was taught in arithmetic textbooks as recently as the 19th century. Here's an example,[1] from *Daboll's Schoolmaster's Assistant*, published in the early 1800s.

A purse of 100 dollars is to be divided among four men A, B, C, and D, so that B may have four dollars more than A, and C eight dollars more than B, and D twice as many as C; what is each one's share of the money?

A modern approach to this would be to let x be the amount given to A. Then B gets $x + 4$, C gets $(x + 4) + 8 = x + 12$, and D gets $2(x + 12)$. Since the total is \$100, we get the equation

$$x + (x + 4) + (x + 12) + 2(x + 12) = 100,$$

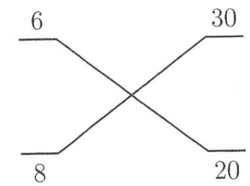
which we then solve in the usual way.

Instead, here's what *Daboll's* recommends: Make a first guess, say that A gets 6 dollars. Then B gets 10, C gets 18 and D gets 36. (Notice that we don't need to work out how D's amount is related to A's to do this; we just go step by step.) Adding up the amounts gives \$70 as the total; we're off by \$30.

[1] Taken from [20], pp. 34–35.

So we try again. This time we guess a little higher, say that A gets 8 dollars. Then B gets 12, C gets 20, and D gets 40, for a total of \$80. That's still wrong, off by \$20.

Now comes the magic. Lay out the two guesses and the two errors as in Display 1. Cross multiply: 6×20 is 120, and 8×30 is 240. Take the difference, $240 - 120 = 120$, and divide by difference of the errors, in this case by 10. The right choice for the amount A gets is $120/10 = 12$.



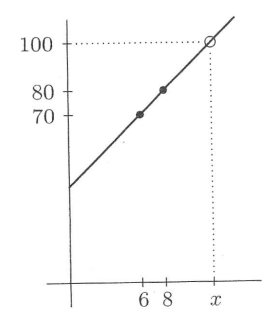
Display 1

This, *Daboll's* explains, is the procedure when the two errors are of the same type (both underestimates, in our case). If they were of different types, we would use the sum of the products and divide by the sum of the errors. (This is just a way of avoiding negative numbers.)

Modern readers usually find this method puzzling: Why does it work? Probably the best way to analyze it is to use some graphical thinking. No matter what the outcome of simplifying the left side of

$$x + (x + 4) + (x + 12) + 2(x + 12) = 100,$$

the equation will be something of the form $mx+b = 100$. So we can think of it like this: there is a line $y = mx + b$, and we would like to determine the value of x for which $y = 100$. To determine the line, we need two points, and the two guesses provide that for us: Both $(6, 70)$ and $(8, 80)$ are on the line. We want to find x so that $(x, 100)$ is on the same line. (See Display 2.) The slope of the line is a constant; we can compute it as "rise over run" using the first and third points. We can also compute it using the second and third points, and the answers must be the same. Therefore, we see that



Display 2

$$\frac{100 - 70}{x - 6} = \frac{100 - 80}{x - 8},$$

or

$$\frac{30}{x - 6} = \frac{20}{x - 8}.$$

Notice that the numerators are exactly the errors we had before. Now cross-multiply to get

$$30(x - 8) = 20(x - 6),$$

which quickly simplifies to

$$(30-20)x = (30\times 8)-(20\times 6);$$

that is,

$$x = \frac{(30\times 8)-(20\times 6)}{30-20} = \frac{120}{10} = 12.$$

This is exactly the same computation as in the method of double false position.

Of course, our way of understanding equations as lines is quite recent (it goes back only to the 17th century; see Sketch 16), and double false position is very old. But the actual slope of the line never needs to be computed. In fact, we don't even have to think of these ratios as slope in any graphical sense. All we need to know is that the change in the output is proportional to the change in the input, which is the essence of what "linearity" is all about. And this the ancients did understand.

The distinction between "linear" and "nonlinear" problems is still useful today. We apply it not only to equations but also to many other kinds of problems. In linear problems, there is a simple relation — a constant ratio — between changes in the input and changes in the output, exactly as we saw above. In nonlinear problems, there is no such simple relation, and sometimes very small changes in the input may produce huge changes in the output. We still don't have a complete understanding of nonlinear problems. In fact, we often use linear problems to find approximate solutions to nonlinear ones. And the methods we use for solving those linear problems are based on the same fundamental insight that serves as the basis for the method of false position.

For a Closer Look: Because solving linear equations is relatively easy, few of the standard history books have sections specifically on that subject. There is a short discussion in [20] (pp. 31–35). Many sample problems can be found in [54].

10 A Square and Things

Quadratic Equations

The word "algebra" comes from a title of a book written in Arabic around the year 825. The author, Muḥammad ibn Mūsa Al-Khwārizmī, was probably born in what is now Uzbekistan. He lived, however, in Baghdad, where the Caliph had established a kind of academy of science called "The House of Wisdom." Al-Khwārizmī was a generalist; he wrote books on geography, astronomy, and mathematics. But his book on algebra is one of his most famous.

Al-Khwārizmī's book starts out with a discussion of quadratic equations. In fact, he considers a specific problem:

> One square, and ten roots of the same, are equal to thirty-nine dirhems. That is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

If we call the unknown x, we might call the "square" x^2. Now, a "root of this square" is x, so "ten roots of the square" is $10x$. Using this notation, the problem translates into solving the equation $x^2 + 10x = 39$. But algebraic symbolism had not been invented yet, so all Al-Khwārizmī could do was to say it in words. In the time-honored tradition of algebra teachers everywhere, he follows the problem with a kind of recipe for its solution, again spelled out in words:

> The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.[1]

Here's the computation in our symbols:

$$x = \sqrt{5^2+39}-5 = \sqrt{25+39}-5 = \sqrt{64}-5 = 8-5 = 3.$$

[1] Translated by Frederic Rosen; see [83], p. 8.

NAME:______________________________ Section:_________

1. What did the English call the study of questions involving unknown numbers?

2. What do you think of Howard Eve's quote?

3. What was Viéte's revolutionary notational device?

4. Describe the method of "false position." (There should be 2-4 sentences for this answer)

5. In the method of "double false position" the author states that we don't even have to compute the actual slope of the line. What does he say we need to know?