Unit 6: Exponents and Radicals

1-1: The Real Number System

Natural Numbers (N): - counting numbers. $\{1, 2, 3, 4, 5, ...\}$

Whole Numbers (W): - counting numbers with 0. $\{0, 1, 2, 3, 4, 5, ...\}$

Integers (*I*): - positive and negative whole numbers. $\{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$

Rational Numbers (Q): - numbers that can be turned into a fraction $\frac{a}{b}$, where $a, b \in I$, and $b \neq 0$.

- include all Terminating or Repeating Decimals.
- include all Natural Numbers, Whole Numbers and Integers.
- include any perfect roots (radicals).
- a) Terminating Decimals: decimals that stops

$$0.25 = \frac{1}{4} \qquad -0.7 = \frac{-7}{10}$$

- **b) Repeating Decimals:** decimals that repeats in a pattern and goes on. $0.3... = \frac{1}{2}$ $-1.\overline{7} = \frac{-16}{9}$
- c) Perfect Roots: radicals when evaluated will result in either Terminating or repeating decimals, or fractions $\frac{a}{b}$, where $a, b \in I$, and $b \neq 0$.

$$\sqrt{0.16} = \pm 0.4$$

$$\sqrt{0.111...} = \pm 0.3... = \pm \frac{1}{3}$$
 $\sqrt{\frac{1}{25}} = \pm \frac{1}{5}$

$$\sqrt{\frac{1}{25}} = \pm \frac{1}{5}$$

$$\sqrt[3]{0.008} = 0.2$$

Irrational Numbers (\overline{Q}): - numbers that **CANNOT** be turned into a fraction $\frac{a}{b}$, where $a, b \in I$, and $b \neq 0$.

- include all non-terminating, non-repeating decimals.
- include any non-perfect roots (radicals).
- a) Non-terminating, Non-repeating Decimals: decimals that do not repeat but go on and on.

$$\pi = 3.141592654...$$

b) Non-Perfect Roots: radicals when evaluated will result in Non-Terminating, Non-Repeating decimals.

$$\sqrt{5} = \pm 2.236067977...$$

$$\sqrt{\frac{1}{6}} = \pm 0.4082482905...$$
 $\sqrt{\frac{1}{6}} = \pm 0.4082482905...$
 $\sqrt{\frac{1}{6}} = -0.7243156443...$

Real Numbers (*R***)**: - any numbers that can be put on a number line.

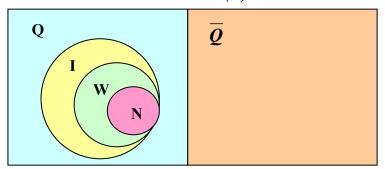
- include all natural numbers, whole numbers, integers, rational and irrational numbers.

Absolute Value |x|: - the positive value of x.

$$|-5| = 5$$
 $|3| = 3$ $|4(-6) + 8| = |-16| = 16$ $|7| - |-1| = 7 - 1 = 6$

In general, we can display the relationships between all types of real numbers in a diagram.

Real Numbers (R)



a) All Natural Numbers belong to the set of Whole Number.

 $N \in W$

b) All Natural and Whole Numbers belong to the set of Integers.

N and $W \in I$

- c) All Natural, Whole Numbers and Integers belong to the set of Rational Numbers. N, W and $I \in Q$
- d) Rational Numbers and Irrational Numbers do NOT belong to each other. (You can have both types at the same time). $Q \notin \overline{Q}$
- e) All Natural, Whole Numbers, Integers, Rational and Irrational Numbers belong to the set of Real Numbers. $N, W, I, Q \text{ and } \overline{Q} \in R$

Example 1: Classify the following numbers.

b. 75

c. -35.24

d. -78.1212...

$$\overline{I,Q}$$
 and \overline{R}

(N, W, I, Q and R)

Q and R

Q and R

e.
$$\sqrt{42}$$

f. 1.459 142 337 ...

g. $\frac{-6}{85}$

h. $\sqrt{\frac{9}{4}}$

$$\overline{Q}$$
 and R

 \overline{Q} and R

Q and R

Q and R

i)
$$\sqrt{0.225}$$

 \overline{Q} and R

j)
$$\sqrt[3]{64}$$

N, W, I, Q and R

=7-5=2

N, W, I, Q and R

$$\begin{vmatrix} -\frac{1}{3} \\ 1 \end{vmatrix}$$

O and R

Inequalities

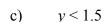
Symbols	Meanings
>	Greater than
<	Less than
≥	Greater than or equal to
≤	Less than or equal to
≠	NOT Equal to
$b_{lower} \le x \le b_{upper}$	x is between the lower and upper boundaries (inclusive).
$b_{lower} < x < b_{upper}$	x is between the lower and upper boundaries (exclusive).
$x \le b_{lower}$ and $x \ge b_{upper}$	x is less than the lower boundary and x is greater than the upper boundary (inclusive).
$x < b_{lower}$ and $x > b_{upper}$	x is less than the lower boundary and x is greater than the upper boundary (exclusive).

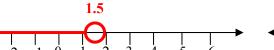
Example 2: Graph the following inequalities.

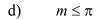






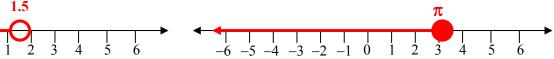




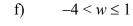


 $x \ge -1$

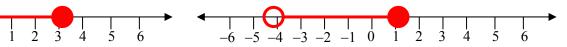
b)

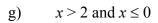


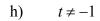
e)
$$-2 \le r \le 3$$







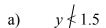


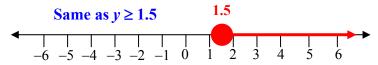




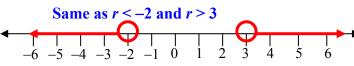


(AP) Example 3: Graph the following inequalities.



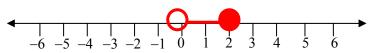


b)
$$-2 \nleq r \nleq 3$$



c)
$$x \not > 2$$
 and $x \not \leq 0$





Example 4: Convert the following decimals to fractions algebraically.

Let x = 0.555 ... (To cancel out the repeating 10x = 5.555... decimals, we have to move the decimal 1 place to the $10x = 5.555 \dots$ right, which means $\times 10$)

$$\underline{- \quad x = 0.555 \dots}$$

$$9x = 5$$

$$x = \frac{5}{9}$$

Let x = 1.3232...(Move the decimal 2 100x = 132.3232... places to the right will

$$100x = 132.3232 \dots$$

$$- x = 1.3232 \dots$$

$$99x = 131$$

$$x = \frac{131}{99}$$

c)
$$-0.264\ 264\ 264\ \dots$$

Let $x = 0.264\ 264\ ...$ $1000x = 264.264\ 264...$

Put the negative sign back!

(Move the decimal 3 places to the right will line up the repeating decimals)

$$\left(-\frac{88}{333}\right)$$

Let x = 3.43535...10x = 34.3535...

$$\begin{array}{r}
 1000x = 3435.3535 \dots \\
 - 10x = 34.3535 \dots \\
 \hline
 990x = 3401
 \end{array}$$

$$x = \frac{3401}{990}$$

First, ignore the negative sign. 1000x = 3435.3535... (Move the decimal 1) place to the right will 1000x = 3435.3535... make the repeating

$$-\frac{3401}{990}$$

Put the negative sign back!

1-1 Homework Assignments

Regular: pg. 8 to 9 #1 to 55, 57, 58, 61, 63, 65 and 66

AP: pg. 8 to 9 #1 to 55, 57 to 59, 61 to 63, 65 and 66

1-3: Evaluating Irrational Numbers

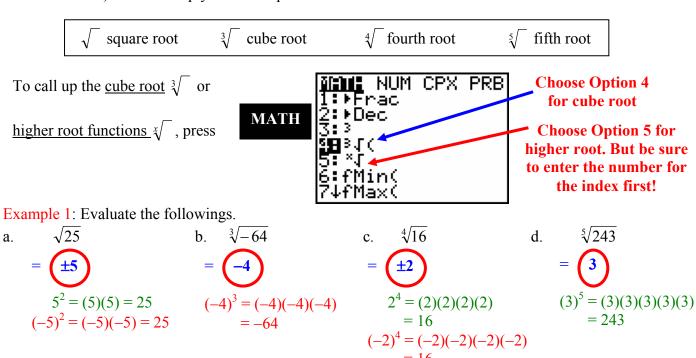
Radicals: - the result of a number after a root operation.

Radical Sign: - the mathematical symbol $\sqrt{\ }$.

radical sign

Radicand: - the number inside a radical sign.

Index: - the small number to the left of the radical sign indicating how many times a number (answer to the radical) has to multiply itself to equal to the radicand.



A radical with an even index always has two answers (±), and can only have a radicand greater than or equal to 0 inside a radical sign.

A radical with an odd index always has one answer only and can have a negative radicand inside the radical sign.

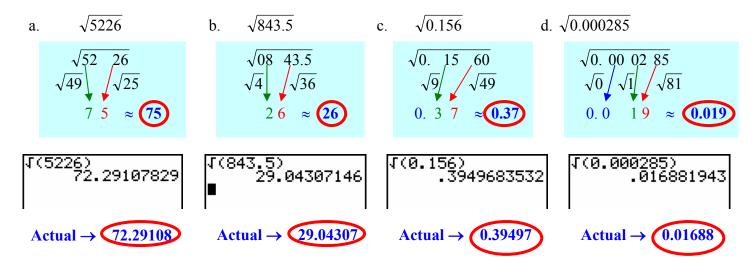
Example 2: A formula $v_f^2 = v_i^2 + 2ad$ can be used to find the final velocity (speed) of an accelerated object, where v_f = final velocity, v_i = initial velocity, a = acceleration, and d = distance travelled. An apple is thrown from the tall building 300 m high with an initial velocity of 6 m/s. The acceleration due to gravity is 9.81 m/s². What is the final velocity of the apple as it reaches the ground?

Solve for
$$v_f$$
:
 $v_f = ?$
 $v_i = 6 \text{ m/s}$
 $d = 300 \text{ m}$
 $a = 9.8 \text{ m/s}^2$
 $v_f = v_i^2 + 2ad$
 $v_f = \sqrt{(6)^2 + 2(9.81)(300)}$
 $v_f = \sqrt{36 + 5886}$
 $v_f = \sqrt{5992}$
 $v_f = \sqrt{5992}$

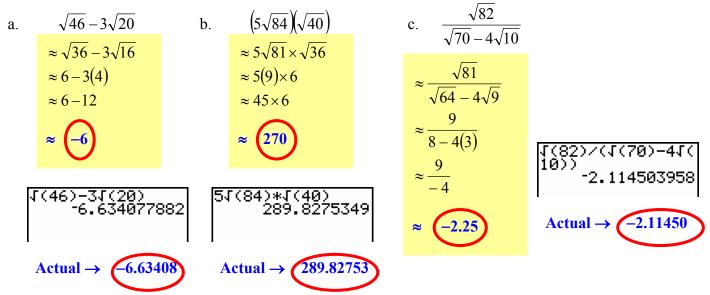
Estimating Square Roots

- 1. Estimating Square Roots GREATER than 1:
 - **a. Group** the radicand by **two digits** starting directly to the <u>LEFT of the decimal place</u>. The digit **0** may be added to the **beginning of the radicand** if there are an odd number of digits.
 - **b.** Estimate each group of two digits by finding the square root of the <u>nearest lower square number</u>.
- 2. Estimating Square Roots LESS than 1:
 - **a. Group** the radicand by **two digits** starting directly to the **RIGHT** of the decimal place. The digit **0** may be added to the **end of the radicand** if there are an odd number of digits.
 - **b.** Estimate each group of two digits by finding the square root of the <u>nearest lower square number</u>.

Example 3: Estimate. Then, find the approximated value to the fifth decimal place using a calculator with only positive roots.



Example 4: Evaluate by estimating, then, find the approximated value to the fifth decimal place using a calculator with only positive roots.



Example 5: Evaluate the followings using only positive roots.

a.
$$\sqrt{36-25}$$

$$= \sqrt{11}$$

$$\approx (3.31662)$$

b.
$$\sqrt{36} - \sqrt{25}$$
$$= 6 - 5$$
$$= \boxed{1}$$

c.
$$\sqrt{36 \times 25}$$
$$= \sqrt{900}$$
$$= \boxed{30}$$

$$\sqrt{36} \times \sqrt{25}$$

$$= 6 \times 5$$

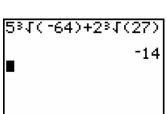
$$= \boxed{30}$$

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \qquad \sqrt{a \times b} = \sqrt{a} \times \sqrt{b}
\sqrt{a-b} \neq \sqrt{a} - \sqrt{b} \qquad \sqrt{a \div b} = \sqrt{a} \div \sqrt{b}$$

Example 6: Evaluate the followings using only positive roots. Verify by using a calculator.

a.
$$5\sqrt[3]{-64} + 2\sqrt[3]{27}$$
$$= 5(-4) + 2(3)$$
$$= -20 + 6$$





b.
$$\sqrt[4]{81} - 7\sqrt[4]{16}$$

= 3 - 7(2)
= 3 - 14



 $\sqrt{\sqrt{0.0256}}$

Example 7: Evaluate the followings using only positive roots. Verify by using a calculator.

a.
$$\sqrt{\sqrt{625}}$$

$$= \sqrt{\left(\sqrt{625}\right)}$$

$$= \sqrt{\left(25\right)}$$

$$= \sqrt{\left(\sqrt{0.0256}\right)}$$
$$= \sqrt{\left(0.16\right)}$$
$$0.4$$

1-3 Homework Assignments

Regular: pg. 14 to 16 #3 to 13 (odd), 14 to 27 (no estimates), 29 to 45, 47 to 52, 54a, 56a and 56b

AP: pg. 14 to 16 #2 to 12 (even), 14 to 27 (no estimates), 29 to 45, 47 to 54a, 56a and 56b

1-4: Simplifying Radicals

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \qquad \text{where } a \ge 0 \text{ and } b \ge 0$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \qquad \text{where } a \ge 0 \text{ and } b > 0$$

Entire Radicals: - radicals that have no coefficient in front of them.

Examples: $\sqrt{52}$ and $\sqrt{48}$

Mixed Radicals: - radicals that have coefficients in front of them.

- the coefficient is the square root of the perfect square factor of the radicand.

Examples: $2\sqrt{13}$ and $4\sqrt{3}$

To convert an entire radical to a mixed radical, find the largest perfect square factor of the radicand and write its root as a coefficient of the remaining radicand factor.

Example 1: Simplify the followings. (Convert them to mixed radicals)

a

$$\sqrt{50}$$

$$=\sqrt{25\times2}$$



 $=\sqrt{25}\times\sqrt{2}$

b.
$$\sqrt{80}$$

$$= \sqrt{16 \times 5}$$
$$= \sqrt{16} \times \sqrt{5}$$

$$4\sqrt{5}$$

$$\frac{\sqrt{108}}{\sqrt{6}}$$

$$= \sqrt{\frac{168}{6}}$$
$$= \sqrt{28}$$
$$= \sqrt{4} \times \sqrt{7}$$

$$2\sqrt{7}$$

$$= \frac{\sqrt{125}}{\sqrt{9}}$$
$$= \frac{\sqrt{25} \times \sqrt{5}}{3}$$



To convert a mixed radical to an entire radical, square the coefficient and multiply it to the radicand.

Example 2: Write the followings as entire radicals.

$$5\sqrt{8}$$

$$= \sqrt{25} \times \sqrt{8}$$
$$= \sqrt{25 \times 8}$$



b.

$$3\sqrt{7}$$

$$= \sqrt{9} \times \sqrt{7}$$
$$= \sqrt{9 \times 7}$$



$$=\sqrt{\frac{4}{9}}\times\sqrt{5}=\sqrt{\frac{4\times5}{9}}$$



Example 3: Order $9\sqrt{2}$, $5\sqrt{3}$, and $4\sqrt{13}$ from least to greatest.

$$9\sqrt{2} = \sqrt{81 \times 2} = \sqrt{162}$$
$$5\sqrt{3} = \sqrt{25 \times 3} = \sqrt{75}$$
$$4\sqrt{13} = \sqrt{16 \times 13} = \sqrt{208}$$

$$\sqrt{75} < \sqrt{162} < \sqrt{208}$$

$$\sqrt{5\sqrt{3}} < 9\sqrt{2} < 4\sqrt{13}$$

Example 4: Simplify.

a.
$$12\sqrt{3} \times 5\sqrt{6}$$

$$= 12 \times 5 \times \sqrt{3} \times \sqrt{6}$$

$$= 60\sqrt{18}$$

$$= 60\sqrt{9} \times \sqrt{2}$$

$$180\sqrt{2}$$

b.
$$(8\sqrt{12})(3\sqrt{8})$$

$$= 8 \times 3 \times \sqrt{12} \times \sqrt{8}$$

$$= 24\sqrt{96}$$

$$= 24\sqrt{16} \times \sqrt{6}$$

$$96\sqrt{6}$$

Rationalization: - turning radical denominator into a natural number denominator by multiplying a fraction of the radical over itself.

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}$$

Example 5: Simplify.

a.
$$\sqrt{\frac{8}{3}}$$

$$= \frac{\sqrt{8}}{\sqrt{3}}$$

$$= \frac{\sqrt{8}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{24}}{3} = \frac{\sqrt{4 \times 6}}{3}$$

$$\frac{2\sqrt{6}}{3}$$

b.
$$\frac{4\sqrt{5}}{\sqrt{6}}$$

$$= \frac{4\sqrt{5}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{4\sqrt{30}}{6}$$

$$= \frac{2\sqrt{30}}{3}$$

c.
$$\frac{3\sqrt{15}}{4\sqrt{5}}$$

$$= \frac{3}{4}\sqrt{\frac{15}{5}}$$

$$= \frac{3}{4}\sqrt{\frac{15}{5}}$$

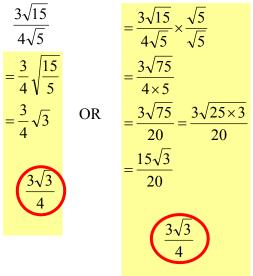
$$= \frac{3\sqrt{75}}{4\times 5}$$

$$= \frac{3\sqrt{75}}{4\times 5}$$

$$= \frac{3\sqrt{75}}{20} = \frac{3\sqrt{75}}{20}$$

$$= \frac{15\sqrt{3}}{20}$$

$$= \frac{3\sqrt{3}}{4}$$



Example 6: Simplify.

$$\sqrt[3]{48}$$

b. 3

$$\sqrt[3]{-250}$$

c. $\frac{\sqrt[3]{7}}{\sqrt[3]{3}}$

d.
$$\sqrt[4]{48}$$

Need to find a perfect cube factor of the radicand.

$$= \sqrt[3]{8 \times 6}$$
$$= \sqrt[3]{8} \times \sqrt[3]{6}$$



We can have a negative perfect cube factor.

$$=\sqrt[3]{-125\times2}$$

$$=\sqrt[3]{-125}\times\sqrt[3]{2}$$

$$-5\sqrt[3]{2}$$

 $= \sqrt[3]{\frac{768}{4}} = \sqrt[3]{192}$ $= \sqrt[3]{64 \times 3}$

$$4\sqrt[3]{3}$$

Need to find a perfect fourth factor of the radicand.

$$= \sqrt[4]{16 \times 3}$$

$$= \sqrt[4]{16} \times \sqrt[4]{3}$$



Example 7: Write the followings as entire radicals.

$$4\sqrt[3]{5}$$

We need to cube the coefficient and multiply it into the radicand.

$$=\sqrt[3]{4^3\times 5}$$

$$=\sqrt[3]{64\times5}$$



$$-5\sqrt[3]{6}$$

$$= \sqrt[3]{(-5)^3 \times 6}$$
$$= \sqrt[3]{-125 \times 6}$$

$$\sqrt[3]{-750}$$

c.
$$\frac{2}{3}\sqrt[3]{10}$$

$$= \sqrt[3]{\left(\frac{2}{3}\right)^3 \times 10}$$
$$= \sqrt[3]{\frac{8}{27} \times 10}$$

$$\sqrt[3]{\frac{80}{27}}$$

d.
$$3\sqrt[4]{8}$$

We need to take the coefficient to the fourth power and multiply it into the radicand.

$$= \sqrt[4]{3^4 \times 8}$$
$$= \sqrt[4]{81 \times 8}$$



(AP) Example 8: Simplify.

$$\sqrt[3]{\frac{7}{6}}$$

$$=\frac{\sqrt[3]{7}}{\sqrt{5}}\times\frac{\sqrt[3]{3}}{\sqrt{5}}$$



We have to multiply the cube root of the square of the radicand to form a perfect cube.

b.
$$\frac{2\sqrt[3]{14}}{\sqrt[3]{9}}$$

$$= \frac{2\sqrt[3]{14}}{\sqrt[3]{9}} \times \frac{\sqrt[3]{81}}{\sqrt[3]{81}} = \frac{2\sqrt[3]{1134}}{9}$$
$$= \frac{2\sqrt[3]{27 \times 42}}{9} = \frac{6\sqrt[3]{42}}{9}$$

$$\underbrace{\frac{2\sqrt[3]{42}}{3}}$$

c. $\frac{3}{2}$

$$= \frac{3\sqrt[3]{98}}{5\sqrt[3]{3}} \times \frac{\sqrt[3]{9}}{\sqrt[3]{9}}$$
$$= \frac{3\sqrt[3]{882}}{5\sqrt[3]{27}} = \frac{3\sqrt[3]{882}}{5\times 3} = \frac{3\sqrt[3]{882}}{15}$$



(AP) Example 9: Solve for x.

$$x\sqrt{42} = \sqrt{7}$$

$$x = \frac{\sqrt{7}}{\sqrt{42}}$$
$$x = \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$\left(x = \frac{\sqrt{6}}{6}\right)$$

$$b. \qquad \frac{5\sqrt{7}}{2x} = 4\sqrt{3}$$

$$\frac{5\sqrt{7}}{2(4\sqrt{3})} = x$$

$$x = \frac{5\sqrt{7}}{8\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$x = \frac{5\sqrt{21}}{24}$$

1-4 Homework Assignments

Regular: pg.19 to 20 #1 to 23 (odd), 25 to 33, 34 to 44 (even), 46 to 55, 59

AP: pg.19 to 20 #2 to 24 (even), 25 to 33, 35 to 45 (odd), 46 to 63

1-5: Operations with Radicals

Adding and Subtracting Radicals:

Convert any entire radicals into mixed radicals first. Then, combine like terms (radicals with the same radicand) by adding or subtracting their coefficients.

Example 1: Simplify the followings.

a.
$$\sqrt{32} - \sqrt{108} + \sqrt{27} - \sqrt{50}$$
$$= 4\sqrt{2} - 6\sqrt{3} + 3\sqrt{3} - 5\sqrt{2}$$
$$= 4\sqrt{2} - 5\sqrt{2} - 6\sqrt{3} + 3\sqrt{3}$$
$$-\sqrt{2} - 3\sqrt{3}$$

b.
$$-3\sqrt{12} + 2\sqrt{20} - \sqrt{75} + 3\sqrt{45}$$
$$= -3(2\sqrt{3}) + 2(2\sqrt{5}) - 5\sqrt{3} + 3(3\sqrt{5})$$
$$= -6\sqrt{3} + 4\sqrt{5} - 5\sqrt{3} + 9\sqrt{5}$$
$$13\sqrt{5} - 11\sqrt{3}$$

Multiplying Radicals:

When multiplying two mixed radicals, multiply the coefficients first, and then multiply the radicands. Simplify each term afterwards if necessary.

Example 2: Simplify the followings.

a.
$$5\sqrt{2}(2\sqrt{3} + \sqrt{8})$$
$$= 5\sqrt{2}(2\sqrt{3} + \sqrt{8})$$
$$= 10\sqrt{6} + 5\sqrt{16}$$
$$= 10\sqrt{6} + 5(4)$$
$$10\sqrt{6} + 20$$

b.
$$2\sqrt{15}(9\sqrt{5} - 7\sqrt{3})$$
$$= 2\sqrt{15}(9\sqrt{5} - 7\sqrt{3})$$
$$= 18\sqrt{75} - 14\sqrt{45}$$
$$= 18(5\sqrt{3}) - 14(3\sqrt{5})$$
$$90\sqrt{3} - 42\sqrt{5}$$

b.
$$2\sqrt{15}(9\sqrt{5} - 7\sqrt{3})$$
 c. $(\sqrt{2} + \sqrt{5})(3\sqrt{2} - 4\sqrt{5})$
 $= 2\sqrt{15}(9\sqrt{5} - 7\sqrt{3})$
 $= 18\sqrt{75} - 14\sqrt{45}$
 $= 18(5\sqrt{3}) - 14(3\sqrt{5})$
 $= 3\sqrt{4} - 4\sqrt{10} + 3\sqrt{10} - 4\sqrt{25}$
 $= 3(2) - \sqrt{10} - 4(5)$
 $= 6 - \sqrt{10} - 20$

d.
$$(4\sqrt{6} - 2\sqrt{3})(7\sqrt{6} + 5\sqrt{3})$$

$$= (4\sqrt{6} - 2\sqrt{3})(7\sqrt{6} + 5\sqrt{3})$$

$$= 28\sqrt{36} + 20\sqrt{18} - 14\sqrt{18} - 10\sqrt{9}$$

$$= 28(6) + 6\sqrt{18} - 10(3)$$

$$= 168 + 6(3\sqrt{2}) - 30$$

$$18\sqrt{2} + 138$$

e.
$$(4\sqrt{3} - 3\sqrt{2})^2$$

$$= (4\sqrt{3} - 3\sqrt{2})(4\sqrt{3} - 3\sqrt{2})$$

$$= 16\sqrt{9} - 12\sqrt{6} - 12\sqrt{6} + 9\sqrt{4}$$

$$= 16(3) - 24\sqrt{6} + 9(2)$$

$$= 48 - 24\sqrt{6} + 18$$

$$-24\sqrt{6} + 66$$

Conjugates: - binomials that have the exact same terms by opposite signs in between.

Example:
$$(a+b)$$
 and $(a-b)$

$$\left(a\sqrt{b} + c\sqrt{d}\right)$$
 and $\left(a\sqrt{b} - c\sqrt{d}\right)$

Multiplying Conjugate Radicals:

The result of multiplying conjugate radicals is **ALWAYS** a <u>Rational Number</u> (the radical terms will always cancel out).

Example 3: Simplify the followings.

a.
$$(\sqrt{5} + 3\sqrt{6})(\sqrt{5} - 3\sqrt{6})$$

$$= (\sqrt{5} + 3\sqrt{6})(\sqrt{5} - 3\sqrt{6})$$

$$= \sqrt{25} - 3\sqrt{30} + 3\sqrt{30} - 9\sqrt{36}$$

$$= 5 - 9(6)$$

=5-54

b.
$$(4\sqrt{7} - 5\sqrt{3})(4\sqrt{7} + 5\sqrt{3})$$

$$= (4\sqrt{7} - 5\sqrt{3})(4\sqrt{7} + 5\sqrt{3})$$

$$= 16\sqrt{49} + 20\sqrt{21} - 20\sqrt{21} - 25\sqrt{9}$$

$$= 16(7) - 25(3)$$

$$= 112 - 75$$

$$37$$

Notice the middle two terms always cancel out!

Rationalizing Binomial Radical Denominators:

Multiply the radical expressions by a fraction consist of the conjugate of the denominator over itself

Example 4: Simplify the followings.

a.
$$\frac{3\sqrt{2}}{\sqrt{5} + 2\sqrt{7}}$$

$$= \frac{3\sqrt{2}}{(\sqrt{5} + 2\sqrt{7})} \times \frac{(\sqrt{5} - 2\sqrt{7})}{(\sqrt{5} - 2\sqrt{7})}$$

$$= \frac{3\sqrt{10} - 6\sqrt{14}}{\sqrt{25} - 4\sqrt{49}}$$

$$= \frac{3\sqrt{10} - 6\sqrt{14}}{5 - 28}$$

$$= \frac{3\sqrt{10} - 6\sqrt{14}}{-23}$$

$$= \frac{-3\sqrt{10} + 6\sqrt{14}}{23}$$

b.
$$\frac{2\sqrt{3} - \sqrt{6}}{3\sqrt{15} - 5\sqrt{2}}$$

$$= \frac{\left(2\sqrt{3} - \sqrt{6}\right)}{\left(3\sqrt{15} - 5\sqrt{2}\right)} \times \frac{\left(3\sqrt{15} + 5\sqrt{2}\right)}{\left(3\sqrt{15} + 5\sqrt{2}\right)}$$

$$= \frac{6\sqrt{45} + 10\sqrt{6} - 3\sqrt{90} - 5\sqrt{12}}{9\sqrt{225} - 25\sqrt{4}}$$

$$= \frac{6\left(3\sqrt{5}\right) + 10\sqrt{6} - 3\left(3\sqrt{10}\right) - 5\left(2\sqrt{3}\right)}{9(15) - 25(2)}$$

$$= \frac{18\sqrt{5} + 10\sqrt{6} - 9\sqrt{10} - 10\sqrt{3}}{135 - 50}$$

$$\frac{18\sqrt{5} + 10\sqrt{6} - 9\sqrt{10} - 10\sqrt{3}}{85}$$

Example 5: A rectangle has a perimeter of $\sqrt{160} + \sqrt{72}$ and its width is $\sqrt{10} - \sqrt{8}$. What is the length of this rectangle?

$$w = \sqrt{10} - \sqrt{8}$$

$$= (\sqrt{10} - 2\sqrt{2})$$

$$P = \sqrt{160} + \sqrt{72}$$

$$= (4\sqrt{10} + 6\sqrt{2})$$

$$P = 2(l+w)$$

$$\frac{P}{2} = l+w$$

$$= (4\sqrt{10} + 6\sqrt{2})$$

$$\frac{P}{2} - w = l$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = 2\sqrt{10} + 3\sqrt{2} - \sqrt{10} + 2\sqrt{2}$$
Switch signs in the second bracket!
$$l = (\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} - 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

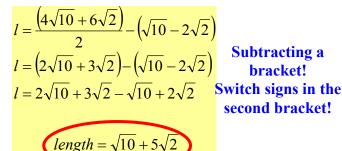
$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

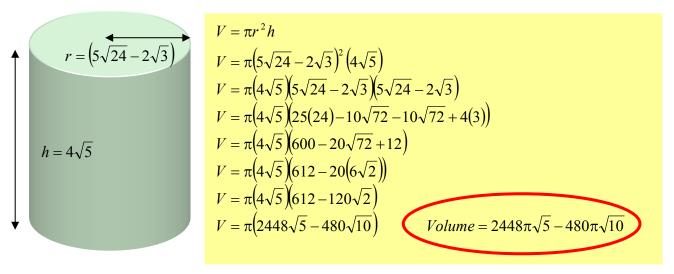
$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

$$l = (2\sqrt{10} + 3\sqrt{2}) - (\sqrt{10} + 2\sqrt{2})$$

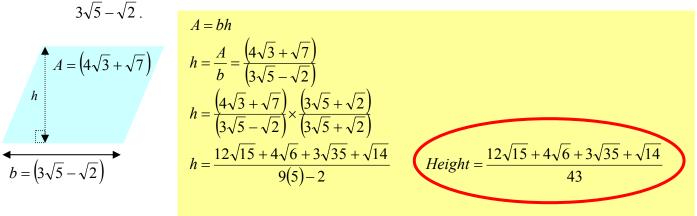
$$l = (2\sqrt{10} + 3\sqrt{2}$$



Example 6: Find the volume of a cylinder that has a radius of $5\sqrt{24} - 2\sqrt{3}$, and its height is $4\sqrt{5}$



Example 7: A parallelogram has an area of $4\sqrt{3} + \sqrt{7}$. Calculate the measure of its height if the base is



Example 8: Simplify.

Example 8: Simplify.
a.
$$\sqrt[3]{128} - \sqrt[3]{16} + \sqrt[3]{250}$$

 $= \sqrt[3]{64 \times 2} - \sqrt[3]{8 \times 2} + \sqrt[3]{125 \times 2}$
 $= 4\sqrt[3]{2} - 2\sqrt[3]{2} + 5\sqrt[3]{2}$

(AP) c.
$$(\sqrt[3]{4} + 5\sqrt[3]{7})(6\sqrt[3]{32} - 2\sqrt[3]{147})$$

= $6\sqrt[3]{128} - 2\sqrt[3]{588} + 30\sqrt[3]{224} - 10\sqrt[3]{1029}$
= $6(4\sqrt[3]{2}) - 2\sqrt[3]{588} + 30(2\sqrt[3]{28}) - 10(7\sqrt[3]{3})$
 $(24\sqrt[3]{2} - 2\sqrt[3]{588} + 60\sqrt[3]{28} - 70\sqrt[3]{3})$

b.
$$2\sqrt[3]{24} - 7\sqrt[3]{81} + 3\sqrt[3]{648}$$

$$= 2\sqrt[3]{8 \times 3} - 7\sqrt[3]{27 \times 3} + 3\sqrt[3]{216 \times 3}$$

$$= 2\left(2\sqrt[3]{3}\right) - 7\left(3\sqrt[3]{3}\right) + 3\left(6\sqrt[3]{3}\right)$$

$$= 4\sqrt[3]{3} - 21\sqrt[3]{3} + 18\sqrt[3]{3}$$

1-5 Homework Assignments

Regular: pg.23 to 24 #1 to 41 (odd), 43 to 60, 64 to 68, 70a

AP: pg.23 to 24 #2 to 40 (even), 43 to 60, 63 to 70

1-7: Reviewing the Exponent Laws

power
$$a \xrightarrow{\text{exponent}} base$$

Exponential Laws

$$(a^{m})(a^{n}) = a^{m+n}$$
 $\frac{a^{m}}{a^{m}} = a^{m-n}$ $(a^{m})^{n} = a^{mn}$ $a^{0} = 1$

$$a^{-n} = \frac{1}{a^{n}}$$
 $(ab)^{m} = a^{m}b^{m}$ $\left(\frac{a}{b}\right)^{m} = \frac{a^{m}}{b^{m}}$

Example 1: Simplify. Express all answers in positive exponents only.

a.
$$(7c^{11}d^4)(-6c^8d^5)$$

$$= -42c^{11+8}d^{4+5}$$

$$-42c^{19}d^{9}$$

b.
$$\frac{9a^3b^3}{-36a^{15}}$$

$$= \frac{1a^{5-15}b^{10-4}}{-4}$$

$$= -\frac{a^{-10}b^{6}}{4}$$

$$= \frac{-b^{6}}{4a^{10}}$$

$$= (3)^3 (x^{5\times3})(y^{2\times3})$$

$$(27x^{15}y^6)$$

d.
$$\frac{\left(5x^3y^2\right)^3\left(3x^5y^9\right)^2}{\left(-6x^7y^3\right)^4}$$
 e.
$$(4m^4n^{-7})^3(2m^3n^5)^{-4}$$

e.
$$(4m^4n^{-7})^3(2m^3n^{-1})^3$$

$$= \frac{(125x^{9}y^{8})(9x^{10}y^{18})}{(1296x^{28}y^{12})}$$

$$= \frac{1125x^{9+10-28}y^{8+18-12}}{1296}$$

$$= \frac{64m^{12}n^{-21}}{16m^{12}n^{20}}$$
when reciprocating an entire bracket, do NOT mess with its
$$= \frac{125x^{-9}y^{14}}{144}$$

$$= 4m^{12-12}n^{-21-20}$$

$$= 4m^{12-12}n^{-21-20}$$

$$= \frac{16p^{-14}q^{-6}}{25p^{-8}q^{6}}$$

$$= \frac{16p^{-14-(-8)}q^{-6}}{25p^{-8}q^{6}}$$

$$= \frac{16p^{-14-(-8)}q^{-6}}{25p^{-8}q^{6}}$$

$$= \frac{16p^{-14-(-8)}q^{-6}}{25p^{-8}q^{6}}$$

f.
$$\left(\frac{-5p^{-4}q^3}{4p^{-7}q^{-3}}\right)^{-2}$$

$$= \left(\frac{4p^{-7}q^{-3}}{-5p^4q^3}\right)^2$$

$$= \frac{16p^{-14}q^{-6}}{25p^{-8}q^6}$$

$$= \frac{16p^{-14-(-8)}q^{-6-(-6)}}{25}$$

$$= \frac{16p^{-6}q^{-12}}{25}$$

$$= \frac{16p^{-6}q^{-12}}{25}$$

g.
$$\frac{3^{-1} - (-3)^2}{(-3)^{-3} + \left(-\frac{1}{3}\right)^{-4}}$$

$$= \frac{\left(\frac{1}{3}\right) - 9}{\left(\frac{-1}{3}\right)^3 + \left(-3\right)^4} = \frac{\frac{1}{3} - 9}{\frac{-1}{27} + 81} = \frac{\left(\frac{-26}{3}\right)}{\left(\frac{2186}{27}\right)}$$
$$= \left(\frac{-26}{3}\right) \div \left(\frac{2186}{27}\right) \qquad \left(\frac{-117}{1093}\right)$$

h.
$$\frac{\left(-6h^{-2}k^3\right)^{-3}}{\left(9h^5k^{-1}\right)^{-2}\left(-3h^{-4}k^{-2}\right)^4}$$

$$= \frac{(9h^{5}k^{-1})^{2}}{(-6h^{-2}k^{3})^{3}(-3h^{-4}k^{-2})^{4}}$$

$$= \frac{81h^{10}k^{-2}}{(-216h^{-6}k^{9})(81h^{-16}k^{-8})}$$

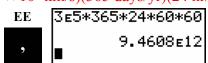
$$= \frac{h^{10-(-6)-(-16)}k^{-2-9-(-8)}}{-216} \qquad \left(\frac{-h^{32}}{216k^{3}}\right)$$

Example 2: In astronomy, one light year is the distance light can travel in one year. Light has a constant speed of 3×10^5 km/s in the vacuum of space.

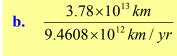
- a. Calculate the distance of one light year.
- b. The closest star to the Sun, Alpha Centuri, is 3.78×10^{13} km. How many light years is it to our sun?

a. One Light Year

 $= (3 \times 10^5 \text{ km/s})(365 \text{ days/yr})(24 \text{ hr/day})(60 \text{ min/hr})(60 \text{ s/min})$









Example 3: Solve for *x*.

a.
$$(x^3)(x^5) = 6561$$

 $x = \sqrt[8]{6561}$

$$(x^3)(x^5) = 6561$$
 b. $x^8 = 6561$

$$(x^3)(x^5) = 6561$$
 b. $\frac{x^2}{x^7} = -1024$

$$x^{2-7} = -1024 x^5 = \frac{1}{-1024}$$

$$x^{-5} = -1024 x = \sqrt[5]{\left(\frac{-1}{1024}\right)}$$

$$\frac{1}{x^5} = -1024 x = \frac{-1}{4}$$

$$\frac{1}{x^5} = -1024$$

(AP) c.
$$\frac{-2x^{-3}}{3x^{-5}} = -96$$

$$x^{-3-(-5)} = -96\left(\frac{3}{-2}\right)$$

$$x^2 = 144$$

$$x = \sqrt{144}$$

$$x = \pm 12$$

(AP) Example 4: Simplify.

a.
$$(-3^{m})^{4} (-3)^{m+4}$$

$$= (-3)^{4m} (-3)^{m+4}$$

$$= (-3)^{4m+(m+4)}$$

$$(-3)^{5m+4}$$

b.
$$\frac{\left(a^{2x}b^{y+1}\right)^3}{a^{3x+1}b^y}$$

$$= \frac{a^{6x}b^{3y+3}}{a^{3x+1}b^{y}}$$

$$= a^{6x-(3x+1)}b^{(3y+3)-y}$$

$$a^{(3x-1)}b^{(2y+3)}$$

1-7 Homework Assignments

Regular: pg.33 #2 to 48 (even), 50 to 82, 84, 85a, 85b, 86

AP: pg.33 #1 to 49 (odd), 50 to 87

1-8: Rational Exponents

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

The index of the radical is the denominator of the fractional exponent.

Example 1: Evaluate.

a.
$$-25^{\frac{1}{2}}$$

b. $(-64)^{\frac{1}{3}}$ $=\sqrt[3]{-64}$

 $(16)^{\frac{3}{2}}$

 $=\sqrt{16^3}$

 $=\sqrt{4096}$

64

d.
$$(-216)^{\frac{-1}{3}}$$

$$= \frac{1}{(-216)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-216}}$$

$$\frac{-1}{6}$$

 $(-27)^{\frac{4}{3}}$

 $=\sqrt[3]{(-27)^4}$

 $=\sqrt[3]{531441}$

f. $\left(\frac{-27}{64}\right)^{\frac{-2}{3}}$

$$= \left(\frac{64}{-27}\right)^{\frac{2}{3}} = \frac{\sqrt[3]{64^2}}{\sqrt[3]{(-27)^2}}$$
$$= \frac{\sqrt[3]{4096}}{\sqrt[3]{729}} \qquad \left(\frac{16}{9}\right)$$

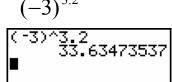
$$= 4^{\frac{-3}{2}}$$

$$= \left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1}{\sqrt{4^3}}$$

$$= \frac{1}{\sqrt{64}} \quad \left(\frac{1}{8}\right)$$

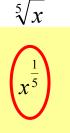
Example 2: Evaluate using a calculator.

a.
$$(-3)^{3.2}$$



Example 3: Write the followings using exponents.

a.



$$\sqrt{2a^3}$$

 $=(2a^3)^{\frac{1}{2}}$

$$= (9y^{12})^{\frac{1}{2}} = 9^{\frac{1}{2}}y^{\frac{12}{2}}$$

$$= \sqrt{9}y^{6}$$

$$= (x^{3}y^{2})^{-\frac{5}{4}}$$

Example 4: Evaluate, if possible.

a.
$$(\sqrt[5]{7^3})(\sqrt[5]{7^2})$$

= $(\sqrt[3]{5})(\sqrt[2]{5})$
= $(\sqrt[3]{5^5})(\sqrt[2]{5^5})$
= $(\sqrt[3]{5^5})(\sqrt[2]{5^5})$

b.
$$\left(5^{\frac{2}{7}} \right)^{\frac{-5}{6}}$$

$$= 5^{\frac{2}{7} \times \frac{-5}{6}}$$

$$\left(5^{\frac{-5}{21}} \right)^{\frac{-5}{21}}$$

c.
$$\left[\left(\sqrt[3]{64} \right)^6 \right]^{\frac{1}{4}}$$

$$= \left[\left(64 \right)^{\frac{6}{3}} \right]^{\frac{1}{4}} = \left[\left(64 \right)^2 \right]^{\frac{1}{4}}$$

$$= 64^{\frac{1}{2}}$$
8

Example 5: Write the following expressions using exponents.

a.
$$\sqrt[4]{256x^9}$$
 b. $\left(27a^{\frac{2}{5}}b^{\frac{-3}{2}}\right)^{\frac{-1}{4}}$ b. $\left(27a^{\frac{2}{5}}b^{\frac{-3}{2}}\right)^{\frac{-1}{4}}$ b. $\left(27a^{\frac{2}{5}}b^{\frac{-3}{2}}\right)^{\frac{-1}{4}}$ b. $\left(27a^{\frac{2}{5}}b^{\frac{-3}{2}}\right)^{\frac{1}{4}}$ b. $\left(27a^{\frac{2}{5}}b^{\frac{-3}{2}}\right)^{\frac{1}{4}}$

a.
$$\sqrt[4]{256x^9}$$
 b. $(27a^5b^2)$ c. $(\sqrt[4]{x})(\sqrt[4]{x})$ d. $(\sqrt[4]{x})(\sqrt[4]{x})$ $= \sqrt[4]{(256x^9)^{\frac{1}{2}}}$ $= \left[(256x^9)^{\frac{1}{2}}\right]^{\frac{1}{4}} = (256x^9)^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{2}}\right]^{\frac{1}{4}} = (256x^9)^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{4}}\right]^{\frac{2}{3}}$ $= \left[(256x^9)^{\frac{1}{2}}\right]^{\frac{1}{4}} = (256x^9)^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{4}}\right]^{\frac{2}{3}}$ $= \left[(256x^9)^{\frac{1}{2}}\right]^{\frac{1}{4}} = (256x^9)^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{2}{3}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$ $= \left[(256x^9)^{\frac{1}{8}}\right]^{\frac{1}{8}}$

c.
$$(\sqrt{x^5})(\sqrt[4]{x^{-3}})$$
 d. $(\sqrt[4]{x^7y^5})^{\frac{2}{3}}$

$$= (x^{\frac{1}{5}})(x^{\frac{-3}{4}})$$

$$= x^{\frac{1}{5} + \frac{-3}{4}}$$

$$= x^{\frac{-11}{20}}$$

$$= x^{\frac{-11}{20}}$$

$$= x^{\frac{11}{20}}$$

$$= x^{\frac{11}{20}}$$

$$= x^{\frac{11}{20}}$$

$$= x^{\frac{11}{20}}$$

$$= x^{\frac{11}{20}}$$

$$= x^{\frac{11}{20}}$$

d.
$$(\sqrt[4]{x^7 y^5})^3$$

$$= \left[(x^7 y^5)^{\frac{1}{4}} \right]^{\frac{2}{3}}$$

$$= (x^7 y^5)^{\frac{1}{4} \times \frac{2}{3}} = (x^7 y^5)^{\frac{1}{6}}$$

$$= x^{7 \times \frac{1}{6}} y^{5 \times \frac{1}{6}} \qquad (x^{\frac{7}{6}} y^{\frac{5}{6}})^{\frac{1}{6}}$$

(AP) Example 6: Solve for x.

a.
$$3^{x} = 243$$
$$3^{x} = 3^{5}$$
$$x = 5$$

$$5^{2x+7} = 125$$

$$5^{2x+7} = 5^{3}$$

$$2x + 7 = 3$$

$$2x = 3 - 7$$

$$2x = -4$$

$$x = \frac{-4}{2}$$

$$x = -2$$

$$2^{3x-5} = \frac{1}{128}$$

$$2^{3x-5} = \frac{1}{2^{7}}$$

$$3x - 5 = -7$$

$$3x = -7 + 5$$

$$3x = -2$$

$$x = \frac{-2}{3}$$

1-8 Homework Assignments

Regular: pg.37 to 38 #1 to 67 (odd), 69 to 83, 84 to 89 (no estimates), 90, 91

AP: pg.37 to 38 #2 to 68 (even), 69 to 83, 84 to 89 (no estimates), 90 to 92